Hacettepe Üniversitesi Eğitim Fakültesi Dergisi

Hacettepe University Journal of Education

e-ISSN: 2536-4758

Examining the Effectiveness of Teaching Through Definitions in the Definition and Conceptualization of Number Sets by Pre-Service Mathematics Teachers*

Ceylan ŞEN, Gürsel GÜLER*****

Article Information

Received: 18.02.2021

Accepted: 20.12.2021

Online First: 30.03.2022

Published: 31.07.2022

ABSTRACT

The purpose of the present study is to identify the effectiveness of the teaching process undertaken by means of definitions on the definition of the mathematical concepts included in number sets by pre-service mathematics teachers as well as their conceptualization concerning the operations performed. The study is designed as a case study, a qualitative research method, and conducted with the participation of 68 pre-service middle-school mathematics teachers selected through purposive sampling. It makes use of individual interviews, worksheets, and videos of teaching process as data collection tools. The study is conducted in two stages: The first stage involves revealing the definitions by pre-service teachers of the mathematical concepts included in number sets with individual interviews in pre-concept teaching through definitions. As for the second stage, the definitions and conceptualization proposed by the pre-service teachers were examined within the framework of worksheets and videos of teaching process during the course entitled "Fundamentals of Mathematics I" taught through definitions. This allows the researcher to identify the effectiveness of definitions through using definitions and conceptualization among pre-service teachers. The findings of the present study led to the conclusion that teaching through definitions is effectiveness in the definition and conceptualization of the concepts included in number sets by pre-service teachers. Furthermore, pre-service teachers are also seen to make definition-based explanations during proving. Definitions are found to be effectiveness in the use of mathematical language and representation by pre-service teachers.

Keywords: Teaching through definitions, definition, conceptualizations, pre-service middle-school mathematics teacher

doi: 10.16986/HUJE.2022.445 Article Type: Research Article

Citation Information: Şen, C., & Güler, G. (2022). Examining the effectiveness of teaching through definitions in the definition and conceptualization of number sets by pre-service mathematics teachers. *Hacettepe University Journal of Education, 37*(3), 969-993. doi: 10.16986/HUJE.2022.445

1. INTRODUCTION

Mathematical definitions form the basis of the significance of the concepts and mathematical proving (Zaslavsky & Shir, 2005). After indicated in the curriculum, the definition of a concept affects the teaching approaches to be adopted by the teachers, the learning process of students, and the solution steps of theorems and proving (Zazkis & Leikin, 2008). Despite the significance of mathematical definitions, they are only developed with the experience of students while learning certain mathematical concepts and they are not usually clearly put forward and discussed within the process of teaching mathematics. However, definitions are crucial while teaching and learning mathematics (Tall & Vinner, 1981; Vinner, 1991). McCrory and Stylianides (2014) found that only 2 out of 16 different mathematics textbooks used within the curriculum of the Mathematics Education programme clearly discuss mathematical definitions and their vital role in mathematics.

Levenson (2012) states that while there is an awareness of the existence of definitions, the body of knowledge on the nature of definitions is inadequate. Wilson (1990) indicates that students experience difficulties in understanding the nature of definitions. Some other studies (e.g., Edwards & Ward, 2004; Vinner, 1991) show that students' performances in terms of proving and reasoning suffer from their inability to understand the roles of mathematical definitions. A study by Ball and Bass (2000) found that primary school students were unable to decide whether the number six is odd, even, or both. The researcher reminded the students to consider the definition of even numbers, leading to the students' realisation that the argument

* This study was found ethically appropriate by the decision of Yozgat Bozok University Instutional Review Board dated 20.01.2021 and numbered E-95799348-050.01.04-3198.

** Assist. Prof. Dr., Yozgat Bozok University, Faculty of Education, Department of Primary Education, Division of Early Childhood Education, Yozgat-Turkey. e-mail: ceylan.sen@yobu.edu.tr (ORCID: 0000-0002-6384-7941)

*** Assoc. Prof. Dr., Yozgat Bozok University, Faculty of Education, Department of Mathematics and Science Education, Division of Mathematics Education, Yozgat-Turkey. e-mail: gursel.guler@yobu.edu.tr (ORCID: 0000-0003-1429-1585)

suggesting that six is an odd number contradicts the definition of even numbers discussed in the class. Similarly, the study by Saxe and colleagues (2013) reported that when teachers sufficiently emphasise the role of mathematical definitions in their teaching processes, primary school students provide explanations based on mathematical definitions in the case where they do not agree with their fellow students' claims and make use of definitions in their arguments established to resolve their disagreements.

Understanding the definitions of mathematical concepts is crucial for teachers as well. As indicated by Ball, Thames, and Phelps (2008), subject content knowledge, which is a component of teachers' knowledge of teaching mathematics, is an important component for teachers to be able to define concepts and know mathematical meaning. Furthermore, Ball (1990) states that teachers must have mathematical knowledge and knowledge about mathematics. Mathematical knowledge contains the knowledge of concepts and operations while knowledge about mathematics includes the nature of mathematical knowledge and conceptions about mathematics as a field (Ball, 1990). For example, the conceptions among teachers regarding the questions of what mathematical definitions are and what their roles in mathematics are constitute examples of the knowledge on the nature of mathematics. Similarly, Edwards and Ward (2004) emphasise that the nature of mathematical definitions must be considered as a concept in its own right and should be known by mathematics students to a certain extent. Seaman and Szydlik (2007) stated that primary school mathematics teachers must have mathematical sophistication and that their understanding of the definitions of concepts constituting the mathematical structure is the prerequisite for this sophistication. Another important aspect of the equipment of teachers with the knowledge of the definitions of mathematical concepts is that the definitions provided in textbooks may not always be correct (Smith et al., 2013) and that in this case, teachers must be able to opt for or develop a useful definition that is mathematical and suitable for the students' level (Ball, 2003). To do so, teachers must have a profound understanding of the definitions regarding mathematical topics and concepts. For instance, they must be knowledgeable regarding the important qualities of mathematical definitions (being suitable for the axiomatic structure, satisfying the required and sufficient conditions) or the question of which qualities are optional for teaching (e.g., elegance, being economical). Another vital point regarding the understanding of the nature of definitions and definitions of mathematical concepts by teachers is that definitions lay the foundations of mathematical knowledge (Mariotti & Fischbein, 1997; Vinner, 1991). Therefore, teachers may benefit from abundant learning opportunities to develop students' skills of abstraction and definition regarding mathematical concepts (de Villiers, 1998).

The ability of mathematics teachers to translate their definition capacities into action depends on a profound understanding of the definition process and their conceptions of the nature of mathematical definitions (Leikin & Winicki-Landman, 2000). In a case study conducted with a high-school mathematics teacher; Johnson, Blume, Shimizu, Graysay and Konnova (2014) demonstrated that the understanding of mathematical definitions by the teacher also affects students' process of concept definition and participation. Another crucial consideration concerning definitions is that mathematical definitions are also effective in the development of mathematical language and communication development in mathematics teaching. As stated by Ball (2003), the conscious use of mathematical language is an important matter in order to understand how definitions shape mathematical problems and what kind of a way to follow during problem-solving and reasoning. The Mathematics Course Curriculum (Turkish Ministry of National Education [MoNE], 2018) identifies learning outcomes concerning explanations and relevant concepts regarding the definition of numbers and operations, geometric shapes, measuring units, and algebraic expressions. Therefore, in order to implement the curriculum, pre-service mathematics teachers must have a sound understanding of mathematical definitions and be able to employ them. It is important to foster the right conceptions regarding mathematical concepts in this respect. The knowledge of teachers concerning the nature of definitions and the definition of mathematical concepts is a vital component of their subject matter knowledge. Borko and Putnam (1996) argued that the subject matter knowledge of teachers may affect their classroom practices as well. Therefore, this reveals the importance of pre-service teachers' knowledge and conception of definitions and of implementation of the activities based on definitions. In addition, Gilboa, Kidron and Dreyfus (2019) stated that pre-service teachers have a little information about how they learn mathematical definitions because they are rarely asked to make definitions. In this direction, it is important to reveal how pre-service teachers define mathematical concepts and the teaching content that supports them to create valid definitions.

Studies on mathematical definitions generally focus on certain mathematical subjects. For example, the studies by Vinner (1977), Dickerson and Pitman (2012), and Levenson (2012) investigated the intellectual processes of undergraduate mathematics students while defining concepts about exponential expressions. Johnson and colleagues (2014) examined how the definitions of mathematical concepts by high-school teachers affect the way students define geometric concepts. The study by Dickerson and Pitman (2016) put forward that undergraduate mathematics students had a hard time coming up with acceptable definitions for irrational numbers. Tirosh and Even (1997) studied the approaches followed by two teachers while solving and defining the operation of $(-8)^{1/3}$, identifying the advantages and shortcomings of these approaches. Some studies concentrate on the definition of geometric concepts (de Villiers, 1998; Mariotti & Fishbein, 1997; van Dorrnolen & Zaslavsky, 2003). Unlike other studies conducted on the subject matter, the present study includes the definition of multiple mathematical concepts, allowing the learning outcomes to be more comprehensive. This comprehensive quality is achieved by the integrated discussion of the concepts of sets, natural numbers, integers, rational numbers, irrational numbers, and real numbers constituting number sets. The study examines and reveals the way pre-service mathematics teachers define and conceptualise the concepts and operations included in number sets. Pre-service teachers are not able to make sense of rational and irrational numbers in number sets (Fischbein, Jehiam & Cohen, 1995; Sirotic & Zaskis, 2007; Toluk-Uçar, 2016) and develop their conceptions based on the natural numbers set (Tirosh, 1998). Similarly, it is also observed that pre-service teachers lack a sound understanding of the meanings of the operations performed using a number or multiple numbers, leading to memorisation (Ball, 1990; Fishbein, Jehiam & Cohen, 1995; Karatas, Guven & Cekmez, 2011; Liljedahl, Sinclair & Zazkis, 2006; Tirosh, 2000; Toluk-Uçar, 2016). As far as number sets are concerned, it is expected from pre-service teachers to make sense of the real numbers set by combining the sets of natural numbers, integers, rational numbers and irrational numbers (Toluk-Uçar, 2016). In this respect, it is important for pre-service mathematics teachers to have conceptual knowledge of numbers and operations performed with numbers.

The present study aims for pre-service mathematics teachers to be able to conceptualise number sets at the end of the teaching process through definitions. With this study, it is aimed to provide a framework for teaching middle school mathematics teacher candidates about the nature and roles of the concepts in number sets, through definitions, and to demonstrate its effectiveness in conceptualising the operations performed. Such frameworks will allow other researchers to make analyses on the mathematical definitions by pre-service middle-school mathematics teachers in the future. Furthermore, teaching concepts through definitions, the effectiveness of which is documented for conceptualisation, would be an available choice for the education of pre-service middle-school mathematics teachers. Therefore, the present study has potential to contribute to the existing body of literature by explaining what mathematical definitions mean for pre-service middle-school mathematics teachers, how they make mathematical definitions, and to what extent mathematical definitions influence the effectiveness of teaching concepts. At the same time, it will also be beneficial for mathematics educators by identifying the effectiveness of definitions in achieving conceptualisation among pre-service teachers during mathematics education.
In this respect, the study attempts to answer the following questions:

1. To what extent the definitions used to teach concepts are effectiveness in the definitions by pre-service teachers of the concepts included within number sets?
2. To what extent the definitions used to teach concepts are effectiveness in the conceptualisation by pre-service teachers of the concepts and operations included within number sets?

1.1. Theoretical Framework

1.1.1. Mathematical definition

Different researchers have varying statements regarding the characteristics required for a mathematical definition to be a good one. Borasi (1992) identifies the qualities of a good definition as concept isolation, essentiality, non-contradiction, and non-circularity. Concept isolation contains exemplary cases regarding the concept and all features of the definition. However, taking one concrete example of the concept would not be sufficient for covering all features of the definition. Essentiality involves including the terms required to distinguish the concept in question from others. Non-contradiction signifies the capacity of involving all statements indicated in the definition in all features of the concept concerned. As for non-circularity, it means not using the term attempted to be defined within the definition itself and a hierarchic structure based on the pre-determined concept. Winicki-Landman and Leikin (2000) dealt with concept teaching within the context of teaching and learning and divides the characteristics of a mathematical definition into two dimensions: a) mathematical features and b) didactic features. Mathematical features signify the logical principles to be included while defining any given concept. As for didactic features, they involve defining a mathematical concept in accordance with the students' level and characteristics. Winicki-Landman and Leikin (2000) described the features constituting a good definition on the basis of studies conducted on mathematical definitions. The first of these is naming for defining. This feature signifies that the definition must include the name of the concept only once. For example, a definition made as "tetragon with four equal sides and angles" defines the shape square but it is not mathematical as it does not contain the term "square". The second feature involves making the definition of a concept only by using a pre-determined concept. The third feature dictates that a definition must satisfy the required and adequate conditions to signify the concept in question. As for the fourth feature, the conditions included in the statement as the definition must be at the minimum level. Finally, definition is an optional preference.

van Dormolen and Zaslavsky (2003) classified and defined the required and preferred properties in definitions as compliance with conceptual hierarchy, existence, equivalence, compliance with the axiomatic structure, the expression of required and sufficient conditions, and an economical characteristic. Compliance with conceptual hierarchy signifies that any novel concept must be a special state of a more general concept (Winicki-Landman & Leikin, 2000). One or more features must be used to describe this special state (van Dormolen & Zaslavsky, 2003). For instance, in the statement "a square is a rectangle with four equal sides", the new concept is "square", the more general concept is "rectangle" and the special characteristic is "four equal sides". Existence signifies the necessity of proving that at least one example of the defined concept must be present within the current context. van Dormolen and Zaslavsky (2003) gives the example of circlesquare and stated that it is unnecessary to make a definition for this concept within Euclidean geometry as it does not exist in Euclidean geometry. The criterion of equivalence requires that in case there are multiple definitions for the same concept, all these definitions must be proven to be equivalent. For instance, there are multiple ways of defining a square. A square can be defined as "a polygon with four equal sides and four equal angles" and "a parallelogram with equal diagonals crossing at right angles". Both statements may be selected as the mathematical definition of the square, but once one is selected, the other will not be a mathematical definition but might be a theorem and it should be proven in accordance with the definition selected (van Dormolen & Zaslavsky, 2003). The criterion of axiomatization states that a definition must be ensured to be suitable to a deductive structure and a part of the system. All concepts used in the definition must be proven to be defined within the same deductive system in that order. This feature is similar to the one defined by Winicki-Landman and Leikin (2000) as suitability to the hierarchic structure, but the emphasis on deductive quality is what makes the two features different. As an example of this criterion, van Dormolen and Zaslavsky (2003) defined a circle as "a figure having the same circularity everywhere". As a circle is not defined in Euclidean geometry, the terms "everywhere" and "circularity" cannot be defined either.

van Dormolen and Zaslavsky (2003) also suggested the criteria of being economical, elegance, and axiomatization for mathematical definitions in addition to the required qualities. The criterion of being economic signifies not mentioning further features of the concept while defining it unless they are necessary. For example, the definition of a square given as "a polygon with four equal sides all four angles of which are 90°" is not an economical definition because there are extra features included. The statement "a square is a polygon with four equal sides and 90-degree angles" fulfills the required and sufficient conditions to define the concept of "square". The criterion of elegance means that in order for a definition to look better, it should require fewer symbols and the use of general basic concepts from which the new concept is derived (van Dormolen & Zaslavsky, 2003; Vinner, 1991; Winicki-Landman & Leikin, 2000). The criterion of axiomatization identified by van Dormolen and Zaslavsky is indicated by Freudenthal (1973) in a similar manner, signifying that considering the linked deductive structures of mathematical definitions, one must know the chain within which the new concept is included and be able to make the connection accordingly (p. 416).

Zaslavsky and Shir (2005) identified obligatory and optional characteristics for a definition. Obligatory characteristics include non-contradiction (having all conditions of a definition together) and clarity, stability in the case of representation changes, and being hierarchical. Unlike other researchers, Zaslavsky and Shir (2005) also put forward optional criteria for definitions that are not obligatory. Being economical is an optional characteristic of definitions. It can be seen that this feature is debatable in terms of some aspects as far as mathematics education is concerned. In one study, Khinchin (1968) puts into question which one of the following definitions for the operation of subtraction is better: "the operation for finding the second addend when the sum and the first addend is given" or "the operation involving removing a number from another number". The researcher stated that the second expression is an explanatory definition and not a mathematical one. Such explanatory definitions allow one to establish an explanatory connection between day-to-day life and experiences based on mathematical definitions (Çakıroğlu, 2013; Khinchin, 1968; Pimm, 1993; Usiskin et al., 1997).

1.1.2. Importance of definitions in mathematics

It is important for learners of mathematics to ascertain the role of definitions in mathematics because one needs to comprehend the role of mathematics and proving to understand mathematics, revealing the epistemological knowledge of learners regarding the functional role of components (Michener, 1978). Students need to require this in order to grasp the mathematics taught to them, which signifies intellectual absence (Harel, 2008). Similarly, teachers must also know why definitions are important in mathematics to actualize their definitive activities. Zaslavsky and Shir (2005) stated that mathematical definitions play four important roles in mathematics. Firstly, definitions are crucial for introducing the components of a theory and for understanding the essence and characteristic features of a concept (Zaslavsky & Shir, 2005). The process of definition is not solely about nomenclature but also about fully characterizing the concept. As explained by de Villiers (1998), "[A definition] is usually accomplished by selecting an appropriate subset of the total set of properties of the concept from which all the other properties can be deduced" (p. 2).

The second important feature of mathematical definitions is that they constitute the fundamental components of a given concept (Zaslavsky & Shir, 2005). Vinner (1991) indicated that there are significant differences between technical concepts (e.g., mathematical concepts) and day-to-day concepts (e.g., house, cat). Vinner (1991) argued that people mostly acquire day-to-day concepts not through definitions but through experience. For instance, one learns about the color red not from its definition but through seeing objects colored red. However, some day-to-day concepts are learned through definitions. For instance, one might define a laptop computer to a friend as "a portable computer one might use on their lap" instead of showing them several laptop computers. In this case, definitions support the creation of conceptual images in one's mind. Concept image is the schemas formed in the mind about that concept (Tall & Vinner, 1981). According to Vinner (1991), after the concept image is created and the establishment of the concept is supported, definitions can be forgotten or become dispensable. In other words, the scaffolding is no longer used after the construction of the building (Vinner, 1991). However, definitions play a vital role in technical concepts. Definitions provide one with the chance of getting oneself rid of mis-conceptualization caused by concept images (Vinner, 1991). Moore (1994) defines conceptualization as making sense of pre-concepts, concepts, and concept images. Regarding internalizing conceptualization, it can be designed in three ways: students can develop their own examples, learn by grasping, and can be designed to grow (Moore, 1994).

Thirdly, mathematical definitions lay the foundations of proving and problem-solving (Zaslavsky & Shir, 2005). Weber (2002) defines proving as a process that "begins with an accepted set of definitions and axioms and concludes with a proposition whose validity is unknown" (p. 14). Moore (1994) examined the difficulties experienced by undergraduate students in terms of proving and reached the conclusion that the limited understanding of the definitions of concepts has made it difficult for students to prove their courses of action.

The fourth important point is that mathematical definitions bring about uniformity for the expression of concepts, allowing for the easier transfer of mathematical ideas (Zaslavsky & Shir, 2005). As stated by Pimm (1993), individuals must agree on the meanings of words in order to communicate with one another. According to Vinner (1991), definitions are inherently inevitable owing to the nature of mathematical concepts. Landau (2001) indicates that definitions and statements included in definitions are created based on expert opinions to establish the communication and validity among those who are familiar with scientific language. Regardless of their use in the past, the expressions are established in a new context on a different basis (Robinson, 1962). For instance, the term "similarity" in geometry has a special meaning and displays distinct features when compared with its habitual use in day-to-day life. Robinson (1962) argued that definitions are advantageous for eliminating uncertainties. In addition to the significant points as far as mathematical definitions are concerned proposed by Zaslavsky and Shir (2005), there is another crucial aspect indicated by Polya (1957). In the present-day, mathematicians are not concerned with the meanings of technical terms but with the mathematical significance put forward by means of mathematical definitions (Polya, 1957). In this case, one might see that definitions have explanatory and descriptive features (de Villiers, 1998; Khinchin, 1968; Polya, 1957).

2. METHODOLOGY

The present study is conducted with the purpose of identifying the effectiveness of concept teaching through definitions (CTD) on the definition and conceptualization of mathematical concepts proposed by pre-service middle-school mathematics teachers (PSTs-M). Based on this purpose, the study aims to examine the processes in which pre-service teachers make definitions and conceptualize knowledge in detail. The studies making in-depth analyses of certain cases, activities and relationships are defined as qualitative research studies (Fraenkel & Wallen, 2009). One of the qualitative research patterns, case studies involve conducting in-depth research on certain phenomena such as the curriculum, individuals, and processes within education (Merriam, 1988). Therefore, the case study design, a qualitative research pattern, is adopted for the present study.

2.1. Research Group

The study was conducted with 68 PSTs-M enrolled at the department of middle-school mathematics education at a state university in Turkey during the fall semester of the 2019-2020 academic year. Among 68 PSTs-M, 52 (76%) were female, while 16 (24%) were male students. For the subject matter to be analyzed well, the selected study group must be able to provide the researcher with data in-detail on the issue researched (Stake, 1995). Therefore, the participants were included in the study through purposive sampling. The enrolment of the participants in the course titled "Fundamentals of Mathematics I" for which the syllabus is planned was taken into consideration during their inclusion in the study. Each pre-service teacher was assigned codes such as P1, P2, …, P68 to protect the privacy of the participants.

In Turkey, the secondary school mathematics teaching program provides four years of education. Graduates of this program teach mathematics in classes from 5th to 8th grade levels of middle schools. All students enrolled in this program are placed with an above average score in the nationwide university entrance exam. For this reason, although most of them are successful students in high school mathematics, their readiness is at a sufficient level. In the first year of the teacher training program (Council of Higher Education [CoHE], 2018), the teaching of basic mathematical concepts and the relationship between the concepts are clearly stated within the scope of the Fundamentals of Mathematics-I course and proving is not explicitly included. Learning the basic mathematical concepts in the Fundamentals of Mathematics-I course is the basis of many field and teaching courses, especially the Number Teaching course in the later periods.

2.2. Research Model

This study was found ethically appropriate by the decision of Yozgat Bozok University Institutional Review Board dated 20.01.2021 and numbered E-95799348-050.01.04-3198. The Fundamentals of Mathematics-I course, the content of which is defined by CoHE (2018), was carried out based on CTD in this study. Designed by the two researchers and implemented by the first author based on CTD during the course entitled "Fundamentals of Mathematics I", the experimental process took 12 weeks. Figure 1 shows the topics and content of the syllabus in-detail. Individual interviews were conducted with each PST-M in pre-concept teaching through definitions (pre-CTD) to reveal the knowledge and definitions among PSTs-M regarding mathematical concepts. Then, CTD was implemented for 12 weeks, and the definition and conceptualization proposed by PSTs-M were examined during these weeks.

The related concepts in the number sets in Figure 1 are discussed in this study. In line with the purpose of the study, firstly, the definition of mathematical concepts and the criteria for definition were presented to PSTs-M. Then, the importance and types of proof in expressing the accuracy and validity of the definitions created by PSTs-M were taught. The concept/s in the number sets were presented to the PSTs-M and they were asked to define the related concept, taking into account the definition criteria of the mathematical concepts taught throughout the process. At this stage, the researcher did not define the related concept, only the PSTs-M were expected to define the concepts. Each statement related to the concept was discussed in detail in the classroom environment.

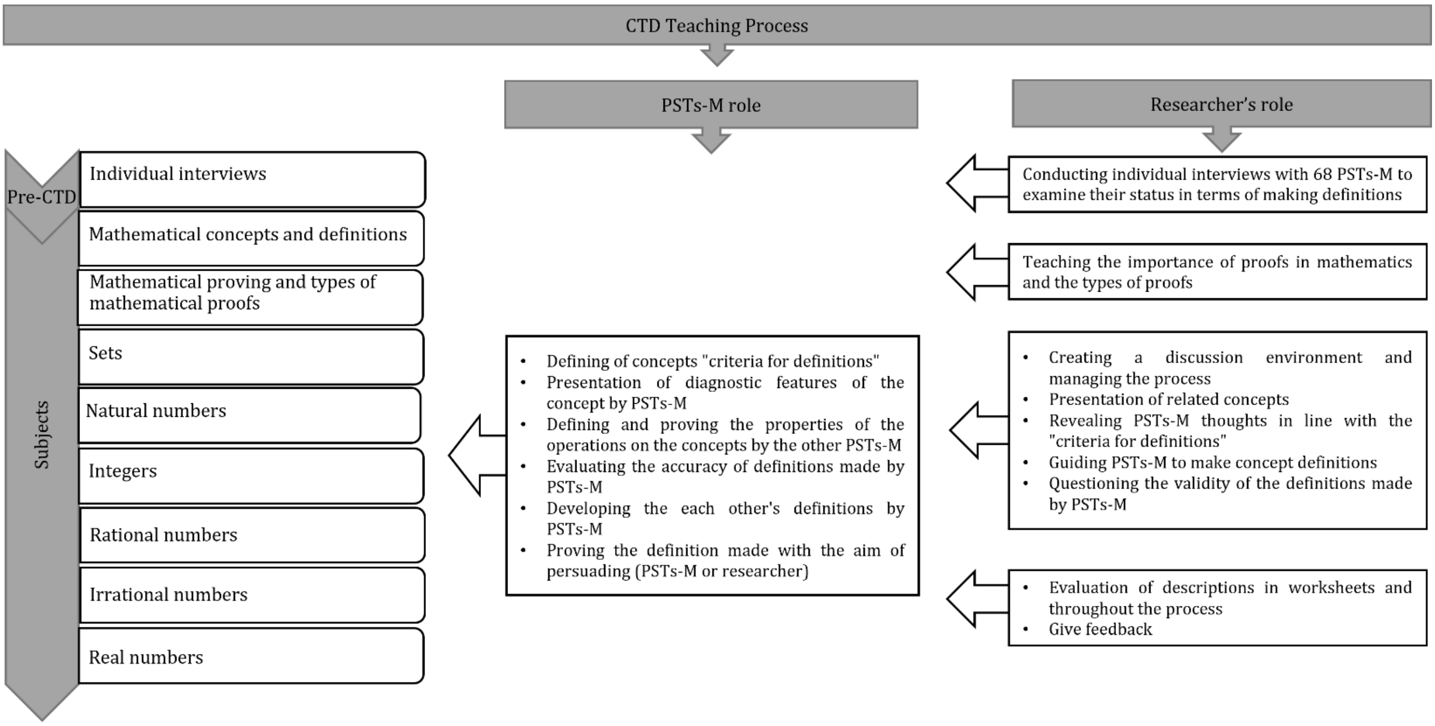


Figure 1. CTD teaching process

During the discussion phase, the researcher guided and asked questions: "What are your descriptive expressions for the concept, according to this definition? Do you think you define the relevant concept in a clear and understandable way? How to edit the description made by your friend?" As a result of the discussion, the definitions of the pre-service teachers regarding the concept presented by the researcher were formed. In this process, the suitability of each statement regarding the concept was evaluated within the scope of definition criteria. It has been tried to imply that there is a need to prove the accuracy and validity of the definitions of PSTs-Ms. They have made the proofs to convince each other and the practitioner. PSTs-M were asked to make definitions of the concepts and to prove their accuracy, and to make sense of the mathematical operations in the subject in this way. For example, after the definition of the integer concept by PSTs-M, it was aimed to define the operations performed on the set of integers and thus to reveal the underlying meaning of the mathematical operations. The concepts included in the subject matter were handled during the CTD teaching process, leading to generalisations made by PSTs-M regarding the definition of the concept in question during discussions. The operation features implemented on the number sets during the process were also examined, leading to the identification of conceptualisation among PSTs-M.

2.3. Data Collection Tools

The reflective interviews conducted between the researcher and students, and student worksheets, and in-class video recordings were used as data collection tools for this study. All class sessions were recorded in order to evaluate the teaching process in an integral manner and to fully reflect the interaction between the instructor and students as well as among students themselves.

2.3.1. Interview

During the pre-CTD stage, semi-structured individual interviews were conducted between the researcher and PSTs-M. A semi-structured interview form was used during these interviews. The interview form consisted of questions prepared in consideration of each concept covered within the syllabus and revealing the definitive, conceptual, and operational knowledge of pre-service teachers. In line with the responses given by the pre-service teachers to the interview questions, their definition and conceptualization situations were examined. Expert opinions of 3 educators who are experts in the field of mathematics education were sought to validate the structure and scope of the prepared interview forms before finalizing and implementing them. The interview form contained 22 questions since it aimed to reveal the knowledge of, and definitions made by pre-service teachers for the concepts of each topic covered within the syllabus. Examples semi-structured sample questions on rational numbers and decimal notation are as follow: "Define what is decimal notation. Can a rational number have two different decimal representations? Please explain. Discuss the accuracy/falseness of the statement "Decimals are fractions". Is the equation true? How did you decide whether the statement is true or false?" The pre-service teachers gave verbal or written explanations to the questions during the interviews. The interviews lasted around 60 minutes with individual sessions for each student.

2.3.2. Worksheets

Worksheets were used in order for students to be able to express the definitions of mathematical concepts both verbally and in writing and to provide explanations and solutions for questions testing their conceptual-operational knowledge throughout the teaching process. The questions in the worksheet were prepared by the researchers in consideration of each subject taking the content scope of the topic in question into consideration. Each worksheet consisted of 10 to 12 questions for PSTs-M to answer individually. Their final versions were formed through expert opinions evaluating the suitability of the questions for the content and scope. While the questions in the worksheets include defining and proving the mathematical concepts in each subject, they consist of questions based on the conceptual dimension of the operations in the relevant subject. In this way, working tools are provided to make the definitions, explanations and operations of PSTs-M.

2.3.3. Teaching process videos

The study was carried out in a face-to-face teaching environment. The teaching process was recorded in order to come up with an account of the teaching process in-detail and to examine the way PSTs-M, in line with the focus of the study, define and conceptualize. For this purpose, all studies of PSTs-M were video recorded and photographed by the researcher. At the same time, audio recording devices were also kept in the teaching environment in order to avoid loss of sound recordings, and in this way, the explanations of each PSTs-M were recorded incompletely. The use of video recordings supported the triangulation of the interviews and worksheets while providing a comprehensive scope in terms of data collection. The video records allowed the researcher to document the process in which they implemented the syllabus, the way pre-service teachers communicate and discuss in the class. All video records were converted into full-text files to prepare them for the analysis.

2.4. Data Analysis

The NVivo 11 software was used for the analysis of the data collected through the methods mentioned above. The data analysis took place in two distinct steps in the consideration of the first and second research questions. Table 1 outlines the data collection tools and the content of the analyses.

Table 1.
Data Collection and Analysis Content

	Data Collection Tools		Analysis Method
Definition processes of PSTs	Interview	Individual interviews	Descriptive Analysis
	Document	PST worksheets	
		Video camera footage	
Concept creation processes of PSTs	Document	PST worksheets	Content Analysis
		Video camera footage	

The voice records of the individual interviews conducted with pre-service teachers as well as the video camera footage regarding the teaching process were converted into written documents. The individual interviews and student worksheets were filed separately for each student. In-class video records were classified as 12 weeks in a way considering the instruction of each concept. The definition processes of pre-service teachers were analysed through descriptive analysis because the criteria for definition were defined by van Dormolen and Zaslavsky (2003). Pre-service teachers' conceptualisations were analysed through content analysis. Content analysis enables data in various forms (audio or video recording, interview, observation, etc.) to be analysed in the most convenient way (Fraenkel & Wallen, 2009). The criteria and categories related to conceptualization are not found in the literature. For this reason, the conceptualization dimension of this study was analysed with content analysis.

2.4.1. Analysis of the definition processes of PSTs-M

Individual interviews, worksheets and video camera footage were used for the analysis of the definition processes among PSTs-M. The analysis of the data was based on the criteria for definitions proposed by van Dormolen and Zaslavsky (2003). Such an analysis of the data within the framework of pre-determined themes is defined as descriptive analysis (Yıldırım & Şimşek, 2016). Pre-CTD individual interviews with pre-service teachers were conducted, revealing their definitions concerning mathematical concepts. Then, their definitions during the CTD stage were presented through worksheets and in-class video recordings. This allowed the researcher to identify the effectiveness of the definitions used for the teaching of mathematical concepts on the definitions made by PSTs-M. Table 2 shows the definition criteria proposed by van Dormolen and Zaslavsky (2003) and the sample codes regarding the definitions by PSTs.

The thematic framework is established through the identification of criteria for definitions. This allowed the researchers to determine the categories according to which the data collected would be coded. The data were coded and processed in line with the explanations in the literature concerning definition criteria (de Villiers, 1998; van Dormolen & Zaslavsky, 2003; Winicki-Landman & Leikin, 2000). The data analysed within the framework of the themes concerned were supported by direct quotations from the information collected during the interviews and from the documents.

Table 2.
Analysis Scheme for the Definition Processes of PSTs-M

Categories		Explanations	Sample coding(s)
Obligatory Features	Defining an existing phenomenon	• Exemplifying the concept defined in a concrete manner	For the sum of positive and negative integers, for example, we have white and black counting scales…
	Equivalence	• Being able to prove that multiple definitions made for the concept are equivalent • Being able to explain the reasons behind the definition selected for the concept • Being able to provide inclusive and exclusive definitions	$Q = \{ \frac{x}{y} \mid x, y \in Z \wedge b \neq 0 \}$ $Q = \{ \frac{a}{b} : a,b \in Z$ Ebob $(a,b) = 1$ ve $b \neq 0 \}$
	Axiomatisation	• Defining each concept on the basis of previously-learned concepts	In case the division of any number within the natural numbers set to another number from the same set does not result in a natural number, I believe we cannot discuss closure; therefore, we obtain the R set in this case.
	Including required and sufficient conditions	• Making definitions by expressing the required and sufficient conditions	The numbers that we cannot write as a rational expression in the form of a/b are irrational numbers.
Optional	Being economical	• Not expressing any conditions other than the ones that are required and sufficient for the concept	Natural numbers are positive integers.
	Elegance	• The definition expressed being clear, creative and thought-provoking	$Z = Z^{-} \cup \{0\} \cup Z^{+}$
	Intuitiveness	• Making definitions without having a deep conceptual understanding regarding the knowledge, rule or concept	Natural numbers are collections of objects, as we always see in school explanations.

2.4.2. Analysis of the conceptualisation processes of PSTs-M

The way the definitions used for concept teaching affecting the conceptualisation processes of PSTs-M were identified by means of worksheets and teaching process video footage. The data collected were analysed through content analysis. The conceptualisation process of PSTs-M was put forward throughout mathematical CTD. An inductive approach was adopted for the analyses regarding the concept creation processes, leading to a four-stage analysis.

The first stage involved dividing the content through which instruction is made for each concept into chapters. This division allowed one to classify the mathematical concepts included in the syllabus. In-class video recordings and worksheets for each mathematical concept were read independently by the researchers, allowing them to come up with codes. Following the creation of the codes by the researchers for the mathematical concepts, the researchers presented the codes they came up with to one another and discussed them during the second stage. During these discussions, they explained them to one another the labels and reasons underlying their codes. The joint assessment of the codes by the researchers resulted in the list of codes for the study. The third stage constituted analysing these codes to establish meaningful relationships between these codes to create categories. The categories at the end of these analyses were “knowledge”, “operation”, “association”, “language”, “reasoning”, and “multiple representation”. As for the fourth stage, the structural relationship between these categories was put into question, leading to the conclusion that the categories of “knowledge”, “association”, “language”, “reasoning”, and “multiple representation” were related to conceptual knowledge while the category “operation” was found to be related to operational knowledge. The themes of conceptual and operational knowledge were obtained at the end of the analysis of the conceptualisation processes among PSTs-M.

2.5. Validity and Reliability

The validity of the study was ensured through data triangulation, expert opinions, purposeful sample selection, and the detailed explanation of the context. Sufficient and required amounts of data were obtained through the use of interviews, video footage, and written documents. This enabled researchers to compare, verify and confirm the data coming from varying sources. The use of purposive sampling and the detailed presentation of the content of the practical aspect in a qualitative study allows it to be applied to similar studies and practices. The presentation of the characteristics of the participants and the outline of the syllabus in-detail and the study process rendered the study transferable.

The reliability of the study was ensured with the separate coding of the data by the two researchers to achieve consistency. To this end, the researchers came up with codes independently before meeting for one last time to establish categories and themes through consensus. Furthermore, data triangulation within the scope of the study aimed to ensure internal consistency.

3. FINDINGS

This section will deal with the definitions made by PSTs-M for mathematical concepts for number sets during the pre-CTD and CTD stages and their conceptualisation processes during the CTD stage.

3.1. Definitions by PSTs-M for the concept of sets during the pre-CTD/CTD phases

Table 3 shows the definitions for the concept of sets and other related concepts given by PSTs-M during the pre-CTD phase identified at the end of the analyses.

Table 3.
Definitions by PSTs-M for the Concept of Sets During the pre-CTD Phase

Concept	Definition	f	%
Set	Collection of a certain number of objects	17	25
	A group of objects	51	75
Proper subset	$2^n - 1$	37	54
	The set apart from itself	31	46
Equal set	Two sets with an equal number of items	48	71
	Two sets with the same items	20	29
Equivalent set	Two sets with an equal number of items	20	20
	Two sets with the same items	48	71
Universal set	The most extensive set	53	78
	The set covering all sets	15	22
Infinite set	Cannot be shown if there are an infinite number of items	58	85
	Does not indicate a set	10	15

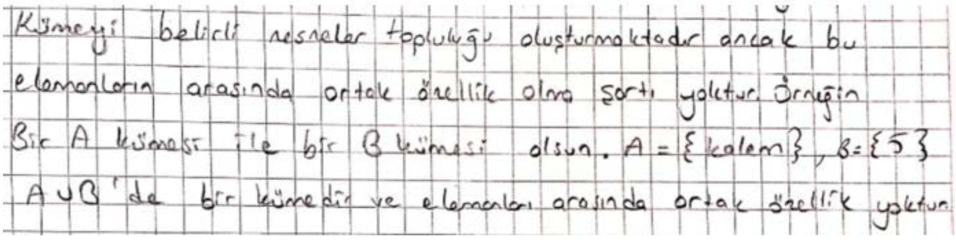
Table 3 shows that PSTs-M made definitions for the concept of sets during the pre-CTD stage such as "*collection of a certain number of objects*" and "*A group of objects*". These statements point out that PSTs-M make intuitive definitions for the concept of sets. At the same time, the expression "*a certain number of*" indicates that they disregard the concept of infinite sets. It was observed that the definitions made by some of the PSTs-M include the term "set", the concept to be defined, in their syntactic compositions. For instance, some PSTs-M defined the concept by making statements such as "*Objects like a, b, c, ... are included within a set named, for instance, A*". Based on this, one might argue that the definitions for the concept of sets provided by PSTs-M are not valid and suitable. They defined the concept of proper subsets operationally as "$2^n - 1$" and as "*The set apart from itself*". These definitions show that pre-service teachers attempt to define the concept with an operational approach and use expressions that are not in line with the definition criteria. The definitions by PSTs-M concerning the concept of universal sets were recorded as "*the most extensive set*", "*a set consisting of many items*", and "*a set including multiple sets, symbolised as E*". The definitions by PSTs-M reveal that the concept of universal sets is defined as sets with many elements or sets. As far as sets with infinite number of items are concerned, PSTs-M generally stated that such sets cannot be expressed due to the infinite number of elements while others claimed that the concept does not indicate a set. When the definitions of the PSTs-M are examined, it is concluded that they do not have an in-depth understanding in this direction and make intuitive definitions. Table 4 presents the definitions for the concept of sets and other related concepts given by PSTs-M during the CTD phase identified at the end of the analyses.

Table 4.
Definitions by PSTs-M for the Concept of Sets During the CTD Phase

Concept	Definition	f	%
Set	Undefined	5	7
	Definition based on the naive and axiomatic set theory	63	93
Proper subset	$2^n - 1$	37	54
	Subsets of a set excluding itself	31	46
Equal set	The set with the same items	68	100
Equivalent set	The set with an equal number of items	68	100
Universal set	The most extensive set worked on	68	100
Infinite set	A set with an infinite number of elements	60	88
	Cardinality	46	68

During the CTD stage, it was observed that PSTs-M made explanations based on the naive and axiomatic set theories, stating that the concept of sets is undefined. The sample definitions by P3 and P46 and the written explanations by P1 (Figure 2) are given below.

> *P3: A good definition of sets relies on the common properties of the elements within the set. However, this indicates that elements without common properties cannot form a set, which is contradictory.*
> *P48: For example, we use the terms "community" or "group" while defining [the concept], but there is no plurality as far as null sets are concerned.*

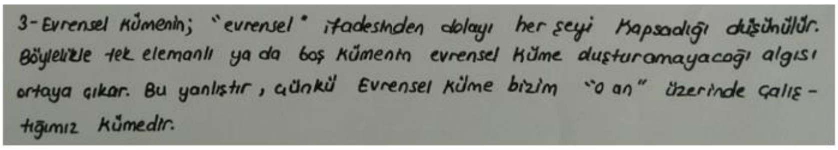


A set is a collection of certain objects, but there is no requirement to be a common feature among these objects. For example, suppose there is a set A and a set B. A={pencil}, B={5}. AUB is also a set and there is no common property among the elements.

Figure 2. Worksheet showing the set definition by P1.

It is seen that the PSTs-M' definitions of set and related concepts are suitable for the axiomatic structure, and they specify necessary and sufficient conditions in their definitions. The sample explanations given by PSTs-M show that they question the terms of "group" and "community" used in definitions and provide examples for contradictory cases. Consequently, they express contradicting examples, proving their points. Figure 3 shows the discussion environment established among the researcher and PSTs-M and sample responses given by PSTs-M.

> ***Researcher (R):** Can a null set be a universal set?*
> ***P7:** The null set is the subset of every set; a universal set cannot be a subset; therefore, a null set cannot be the universal set*
> ***R:** Do you agree with the statement of your colleague?*
> ***P16:** No, because we define the universal set as "the most extensive set we work on". Therefore, if the most extensive set we work on is a null set, the null set can be the universal set.*

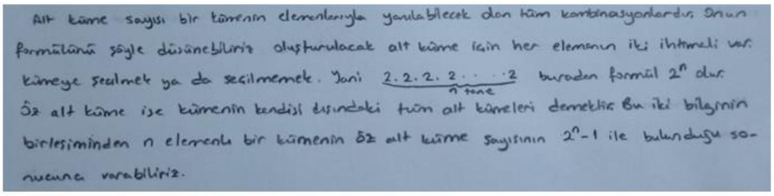


The universal set is considered to be all-encompassing because of the expression "universal". Thus, it is thought that a single-element or empty set cannot form a universal set. This is wrong because the universal set is the set we are "at moment" working on.

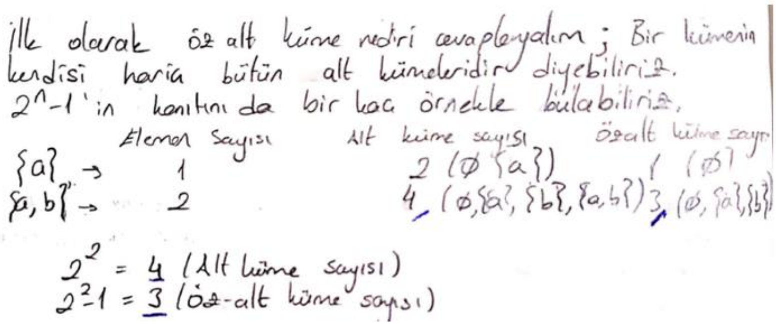
Figure 3. Worksheet showing the universal set definition by P9.

The explanations made by pre-service teachers concerning the question of whether a null set can be the universal set reveals that their explanations made to convince one another are based on definitions. Similarly, the question of "*are equivalent sets also equal sets?*" was asked to pre-service teachers and they argued that this is not true based on definitions. The sample explanation by P9 is as follows: "*Equality signifies that the elements of the set are the same while equivalence denotes an equal number of elements. Therefore, we cannot make a generalisation arguing that two sets must contain the same elements if they have the same number of elements, they may have different elements*". During the teaching of sets, the concept of proper subsets was introduced with the discussion of properties of sets; then, pre-service teachers were asked why the number of proper subsets is $2^n - 1$. The explanations made by P29 and P42 in response to this question are given in Figure 4 as direct quotations.

The PSTs-M first made assumptions based on their definition of the number of subsets and then proved their assumption. Based on these solutions and explanations, one can see that PSTs-M make justifications based on definitions while proving their points and attempting to convince their colleagues. In this direction, it is concluded that PSTs-M prefer to prove the validity of their definitions and create definitions suitable for the axiomatic structure. After the exemplification of different representations of sets and number sets (natural numbers, integers, rational numbers, irrational numbers, and real numbers), a discussion environment was established on the subject of finite and infinite sets. Within this process, the explanations by P5 and P17 concerning finite and infinite sets were as follows: "*When we talk about a set like A = (0,1,2,3,4,5), it is a finite set; but when we try to define natural numbers, the expression N =* $\{0,1,2,3,\ldots\}$ *consists of infinite number of elements*" and "*both natural numbers and real numbers form infinite sets, but the cardinalities of these sets are different*". Based on these results, one can see that PSTs-M adopt a conceptual approach concerning the definition of the concept of sets. The explanations by PSTs-M indicate that the mathematical meanings of the concepts of sets, equal and equivalent sets, finite and infinite sets and universal sets are structured, and the conceptualisation of these concepts is successful.



A subset number is any combination of elements of a set that can be written. We can think of the formula as follows. Each element for the subset has two possibilities: to be selected or not to be selected. 2.2.2.....2 (n) so the formula is 2^n.
A core subset is all subsets of the set except itself. From the combination of these two information, we can conclude that the core subset number of a set is $2^n - 1$.



Let's first answer the question "what is a core subset"; we can say that all subsets of a set except itself. $2^n - 1$ proof is in the image.

Figure 4. Sample worksheets of P29 and P42 concerning the definition of proper subsets.

3.2. Definitions by PSTs-M for the concept of natural numbers during the pre-CTD/CTD phases

The definitions given by PSTs-M during the pre-CTD stage concerning the concept of natural numbers and the properties of operations performed with natural numbers were examined. Table 5 shows these definitions.

Table 5.
Definitions by PSTs-M for the Concept of Natural Numbers During the pre-CTD Phase

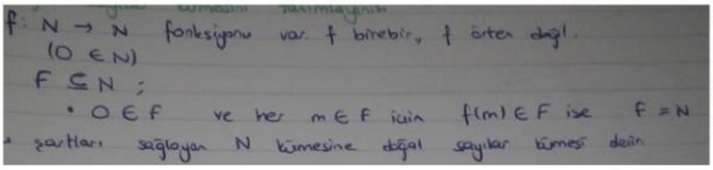
Concept	Definition	f	%
Natural number	Positive integers starting from 0 to plus infinity	64	95
	Numbers starting from 1 until infinity	1	1
	The set of positive integers	1	1
	Counting numbers	2	3
Subtraction not displaying the closure property	Undefined	58	85
	Resulting negative integers	10	15
Division not displaying the closure property	Undefined	57	84
	Emergence of the rational numbers set	11	16

Most PSTs-M defined natural numbers as “*positive integers starting from 0 and incrementing to infinity*” during the pre-CTD stage. One can see that PSTs-M make their definitions concerning natural numbers based on integers. Similarly, the definitions by PSTs-M include explanations such as “*the natural numbers set is a subset of the integers set; therefore, one can call natural numbers of positive integers*”. Based on these definitions, one reaches the conclusion that PSTs-M make their definitions concerning natural numbers on the basis of the integers set and therefore, these definitions are not axiomatic. Concerning the operations performed using natural numbers, PSTs-M were also asked about the results of the operation of subtraction not displaying the closure property; most PSTs-M stated that they are not familiar with the closure property. Besides these responses, P22 and P35 expressed the need for defining negative integers for the fact that subtraction does not display the closure property by making explanations as follows: “*[the subtraction of natural numbers] may result in negative integers*” and “*we do not always get a natural number upon subtracting one natural number from another. Therefore, the operation does not display the closure property*”. As far as the non-closure of division in natural numbers is concerned, most PSTs-M were not able to define this situation while some stated that this results in the definition of rational numbers. Concerning this situation, P47 made the following explanation: “*In case the division of any number within the natural numbers set to another number from the same set does not result in a natural number, I believe we cannot discuss closure; therefore, we obtain the R set in this case*”. It was observed during the pre-CTD stage that PSTs-M did not indicate the required and sufficient conditions in their definitions for natural numbers, these definitions do not satisfy the criteria of axiomatisation and equivalence, and the level of conceptual knowledge among the participants concerning the operations on natural numbers is not adequate. Table 6 shows the definitions by PSTs-M regarding natural numbers and the operations with natural numbers during the CTD stage.

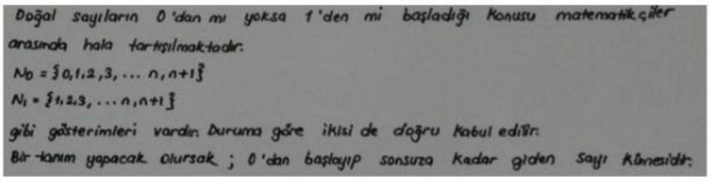
The definitions made by PSTs-M regarding the concept of natural numbers show that they are based on the Peano axioms and that the starting point is either 0 or 1. The sample definition for this by P2 was as follows: “*The set that consists of numbers starting from 0 or 1 and incrementing to infinity*”. In addition to these definitions, Figure 5 shows the sample definitions made by P3, P16 and P17 based on the Peano axioms.

Table 6.
Definitions by PSTs-M for the Concept of Natural Numbers During the CTD Phase

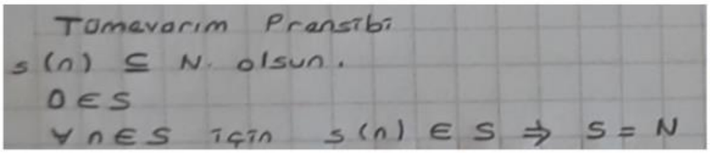
Concept	Definition	f	%
Natural number	Start from 0	16	23
	Start from 0 or 1	52	77
Operations with natural numbers	Modelling	47	69
	Representation with symbols	21	31
Subtraction not displaying the closure property	Definitions based on the identity element	10	15
	Defining negative numbers	58	85
Division not displaying the closure property	Definitions based on the identity element	10	15
	Defining rational numbers	58	85



There is f: N→N function. f is one-to-one, f is not surjective.
(0∈N)
F⊆N
0∈F and for each m∈F which satisfies the conditions f(m) ∈F are called the set of natural numbers N set.



Whether natural numbers start from 0 or 1 is still debated by mathematicians.
There are indications such as;
$N_0 = \{0,1,2,3,\ldots,n,n+1\}$
$N_1 = \{1,2,3,\ldots,n,n+1\}$
If we make a definition; the set of natural numbers is the set of numbers starting from 0 to infinity.



Inductive Method
Let s(n)⊆N
0∈S
s(n) ∈S for ∀n∈S⇒S=N

Figure 5. Worksheets showing the natural number definitions by P3, P16 and P17.

The use of axioms and the principle of induction by PSTs-M while defining the concept of natural numbers indicates the satisfaction of the criteria of axiomatisation and elegance in their definitions. At the same time, the definitions for the natural numbers set through different expressions also fulfil the criteria of equivalence and the satisfaction of required and sufficient conditions. The indication of axioms for the definition of natural numbers shows that the mathematical thought and understanding is developed regarding the question of why this set can begin with either 0 or 1. Consequently, the conceptual approach towards natural numbers among PSTs-M is seen to be achieved.

They used the numerical axis and area modelling to define and explain the operations on natural numbers. To this end, they were asked to solve the mathematical operations of 3 + 8 = 11, 8 – 3 = 5, and 5 × (4 + 7). In this respect, they seem to model these operations based on the definitions of the operations of addition, subtraction, and multiplication (Figure 6).

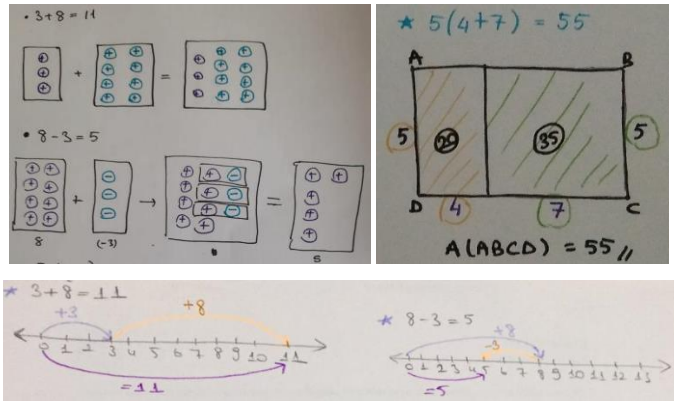


Figure 6. Natural number operation models created by PSTs-M.

Instead of merely using the algorithm, PSTs-M were observed to make explanations requiring a conceptual basis for their solutions for operations made with natural numbers. Similarly, P56 explained the non-closure of subtraction and division for natural numbers based on definitions and made the following statement: "*As there is no identity element in subtraction, the disclosure property is not there for subtraction and division*". Additionally, P11 made the following generalisation after the discussion of the properties of the natural numbers set: "*Each number set is defined based on natural numbers*". Considering these findings, the definitions made by PSTs-M were seen to be effectiveness during proving and generalisations.

3.3. Definitions by PSTs-M for the concept of integers during the pre-CTD/CTD phases

The definitions given by PSTs-M during the pre-CTD stage concerning the concept of integers and the properties of operations performed with integers were examined; Table 7 shows these definitions.

Table 7.
Definitions by PSTs-M for the Concept of Integers During the pre-CTD Phase

Concept	**Definition**	**f**	**%**
Integer	Combination of positive and negative integers	16	24
	Combination of positive and negative integers including 0	14	20
	Numbers from -∞ until +∞	38	56
(-)	Subtraction	45	66
	The number being negative	23	33
(+)	Addition	45	66
	The number being positive	23	33

The examination of the definitions by PSTs-M concerning the concept of integers showed that they generally define the concept as "*numbers constituting the combination of positive and negative integers*". Based on this definition, it seems like PSTs-M do not include the number 0 in their definition. However, some PSTs-M include 0 in this definition, expressing it as "*the set consisting of positive and negative integers and 0*". In these definitions, PSTs-M seem to include the term "integer", the concept to be defined, in their definition sentences. Therefore, the definitions are not considered to be suitable. Additionally, some PSTs-M defined integers as "*numbers ranging from -∞ to +∞*" without analysing the in-depth meaning of this concept. One might argue based on the definitions for the concept of integers that PSTs-M make intuitive definitions concerning the concept asked in the question.

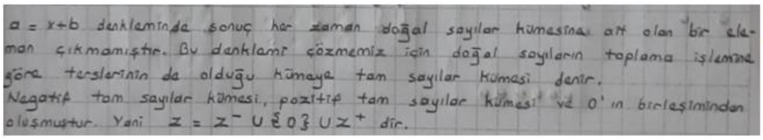
Upon being asked what the symbols (-) and (+) seen next to integers signify, PSTs-M said (-) means either "*the symbol indicating that the number is negative*" or "*subtraction*". Similarly, they also defined the operator (+) as either "*the symbol indicating that the number is positive*" or "*the operation of addition*". Following these explanations, some sample operations with integers were presented to PSTs-M and they were asked to indicate the meanings of the symbols (-) and (+). For the symbol (-) in the operation (+2)-(-6), P33 said that it signifies "*subtracting a negative number from a positive number*". Making similar explanations, PSTs-M identified this as the operation of subtraction. In line with these explanations, some PSTs-M appear to refer to the operation based on the definition of the concept. Certain PSTs-M, on the other hand, indicated that "*-(-6) means +6, therefore (+2)+6=8, meaning [that we do] addition*". In light of this explanation, one can infer that PSTs-M make explanations based on operations. They were given the mathematical expression (-2)-(-6), which was defined as subtraction.

The operation of subtraction indicated for this expression was defined in a variety of ways by PSTs-M. For instance, the pre-service teachers P48 and P66 made the following explanations: "*there are two negative numbers here, for example, the subtraction of (-6) from (-2)*" and "*here, two (-) signs become (+) when they are side by side, so the operation becomes (-2)+6, subtraction of 2 from 6*". According to these statements, one can observe that some pre-service teachers make conceptual explanations whereas some others base their explanations on operational meaning. Table 8 shows the definitions by PSTs-M regarding integers and the operations with natural numbers during the CTD stage.

Table 8.
Definitions by PSTs-M for the Concept of Integers During the CTD Phase

Concept	**Definition**	**f**	**%**
Integer	Definition based on the natural numbers set	51	75
	Definition with mathematical symbols	23	25
Meanings of the symbols (-) and (+)	Operation	68	100
	Direction	68	100

The examination of the way PSTs-M define integers during the CTD stage shows that the definitions are based on the natural numbers set and mathematical representations. A sample explanation by P8 concerning the definition of integers on the basis of natural numbers reads as follows: "*integers have emerged as a result of the lack of closure in subtraction for natural numbers. All numbers between -∞ and +∞, including 0, are within this set*". A similar definition by P62 using mathematical symbols is shown in Figure 7.



In the equation a=x+b, the result is not always an element that belongs to the set of natural numbers. In order to solve this equation, the set of natural numbers with their inverse according to the addition operation is called the set of integers. It is a combination of negative integers, positive integers, and 0. So; $Z = Z^{-}U\{0\}UZ^{+}$

Figure 7. Sample definition for integers made by P62.

Looking at the example explanations, it is seen that PSTs-M define integers based on the set of natural numbers. In addition, it is seen that the definition of P62 is expressed in an economical and aesthetic way, one of the criteria of being a good definition, by using mathematical symbols. In line with these results, PSTs-M are seen to define integers in an axiomatic and economical manner satisfying required and sufficient conditions. A discussion environment was established concerning the (-) and (+) symbols included in operations using integers. The sample explanations and models given by PSTs-M during this discussion process are presented below (Figures 8-9).

> ***P10:*** *The numbers have either the (+) or (-) sign in front of them based on their positions relative to 0, which we call positivity or negativity. I will use counting pieces to demonstrate their positions relative to 0. The red pieces represent positive numbers while the green ones are negative numbers.*
> ***R:*** *How would you define the operation (-2)-(-6)?*
> ***P44:*** *We first need to see whether the signs show an operation or a direction. The signs in (-2) and (-3) are directions while the (-) in between is the operation. It is easier to show these directions and operations on the numerical axis. First, we took 2 steps to the left of 0; then, we took 6 steps in the opposite direction. Therefore, we arrived at 4, on the right of 0, giving us the result of (+4).*

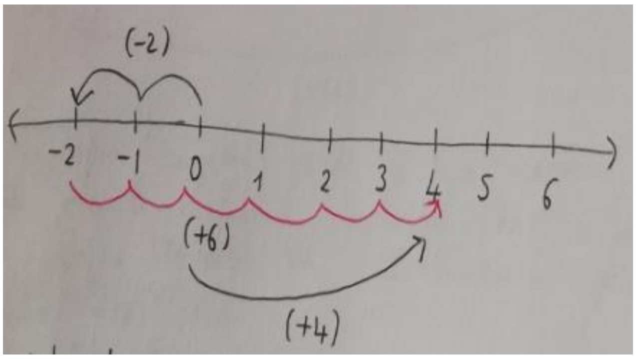
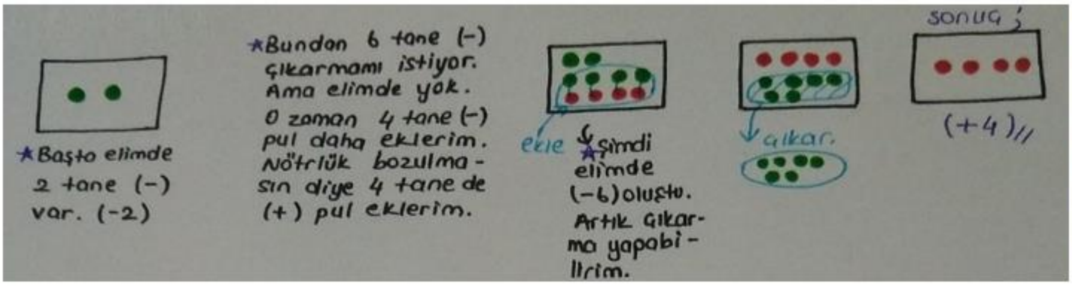


Figure 8. Numerical axis modelling for operations with integers.

> ***P51:*** *I agree, and this is how we do the operation with counting pieces. In the beginning, I have 2 green pieces indicating the number (-2). In this case, the (-) sign in between denotes the operation, meaning that I have to do subtraction. (-6), so we need to subtract 6 green pieces, which we do not have. Therefore, I will add 4 green pieces in addition to 4 red (positive) pieces neutralising these additions. In the end, 4 red pieces remain after the other green and red pieces neutralise one another, leading to the result of (+4).*



We have 2 (-) at first. We are asked to subtract 6 (-) from it. But we don't have it. Then I add 4 more (-) stamps. I add 4 (+) to keep the neutrality. Now we have (-6). I can take it out now. Result: (+4)

Figure 9. Counting piece modelling for operations with integers.

PSTs-M were observed to use the numerical axis and counting pieces while defining the written signs (-) and (+) as either directions or operations as far as integers are concerned. Based on this, one might argue that CTD contributes to the conceptualisation of integers.

3.4. Definitions by PSTs-M for the concept of rational numbers during the pre-CTD/CTD phases

The definitions given by PSTs-M during the pre-CTD stage concerning the concept of rational numbers and the properties of operations performed with rational numbers were examined; Table 9 shows these definitions.

Table 9.
Definitions by PSTs-M for the Concept of Rational Numbers During the pre-CTD Phase

Concept	Definition	f	%
Rational number	Fractional numbers	3	4
	Numbers that can be written as a/b	45	66
	Numbers that can be written as a/b, with b≠0	20	29
Addition	Expression as an operation	68	100
Subtraction	Expression as an operation	68	100
Multiplication	Expression as an operation	68	100
Division	Expression as a formula	68	100

The definitions given by PSTs-M for rational numbers in the pre-CTD stage include expressions like "*fractional numbers*" and "*numbers that can be written as a/b, with b≠0*". One might see from these definitions that PSTs-M have a hard time differentiating rational numbers and fractions, consequently leading to false definitions. Sample definitions by PSTs-M include statements such as "*expressions like a/b that cannot be written as unit fractions*" and "*fractional numbers written as a/b*". The definitions and explanations provided by PSTs-M indicate that they are not able to define rational numbers correctly.

PSTs-M explained the addition and subtraction operations with rational numbers with an operational approach during the pre-CTD stage. For addition and subtraction with rational numbers, PSTs-M provided the following explanation: "*if the denominators are equal, the sum of the nominators are written to the nominator part; if the denominators are not equal, we make the denominators same before addition or subtraction*". Similarly, multiplication is defined with the stages involved in the operation as "*the nominators are multiplied to find the nominator of the result and the denominators are multiplied to find the denominator*". As for division, it is defined in an algorithmic manner as "*we keep the first fraction as it is, then we turn the second fraction upside down the second fraction and make multiplication*". Based on the explanations by PSTs-M, it can be seen that they describe the stages of operations instead of defining the operations themselves. In line with this result, one might argue that PSTs-M do not have sufficient knowledge to define rational numbers and the operations using rational numbers and are unable to differentiate rational numbers from fractions and accordingly, they intuitively define the concept of rational number. Rational numbers were taught during the CTD stage; Table 10 shows the definitions made by PSTs-M during this stage for rational numbers and operations made with rational numbers.

Table 10.
Definitions by PSTs-M for the Concept of Rational Numbers During the CTD Phase

Concept	Definition	f	%
Rational number	Multiplicative inverses of integers	11	16
	Numbers that can be written as a/b, with b≠0 and a and b being coprime numbers	57	84
Addition	Conceptual expression	68	100
Subtraction	Conceptual expression	68	100
Multiplication	Conceptual expression	68	100
Division	Conceptual expression	53	78
	Expression as multiplication	15	22

Most PSTs-M defined rational numbers during the CTD stage as "*numbers that can be written as a/b, with b≠0 and a and b being coprime numbers*". With this definition, one can see that it satisfies the required and sufficient conditions for defining rational numbers. Furthermore, P37 and P64 were also seen to define their statements using mathematical symbols (Figure 10).

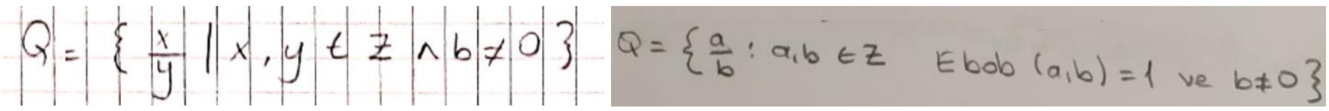


Figure 10. Sample rational number definitions by P37 and P64.

One might argue that the definitions made by PSTs-M using mathematical symbols fulfil the criteria of being economical and elegance. On the other hand, some PSTs-M expressed this definition considering equation solving as "*with a and b being integers and b≠0, the solution of the equations like bx=a is a/b, and these indicate rational numbers*". This way of making the definition demonstrates suitability to conceptual hierarchy. PSTs-M make definitions using various ways concerning the concept of rational numbers, arguing that these definitions are equivalent and correspond to the concept of rational numbers. Following the definition of rational numbers, a discussion setting was established regarding the difference between fractions and rational numbers. Some exemplary explanations by PSTs-M were as follows: "*both can be written as a/b but fractions cannot be negative*" and "*we speak of the condition of a and b being integers for a rational number of a/b; but in fractions, a and b must be natural numbers*". Based on this, PSTs-M are seen to make definition-based justifications while trying to convince others and provide explanations.

They used models in line with the axiomatic structure in their implementations and explanations on addition and subtraction with rational numbers during the CTD stage. PSTs-M were given the operation $\frac{2}{5}+\frac{3}{4}$ and asked to elaborate on this operation. Figure 11 shows the sample models given by P6, P42 and P51 for the operation in question.

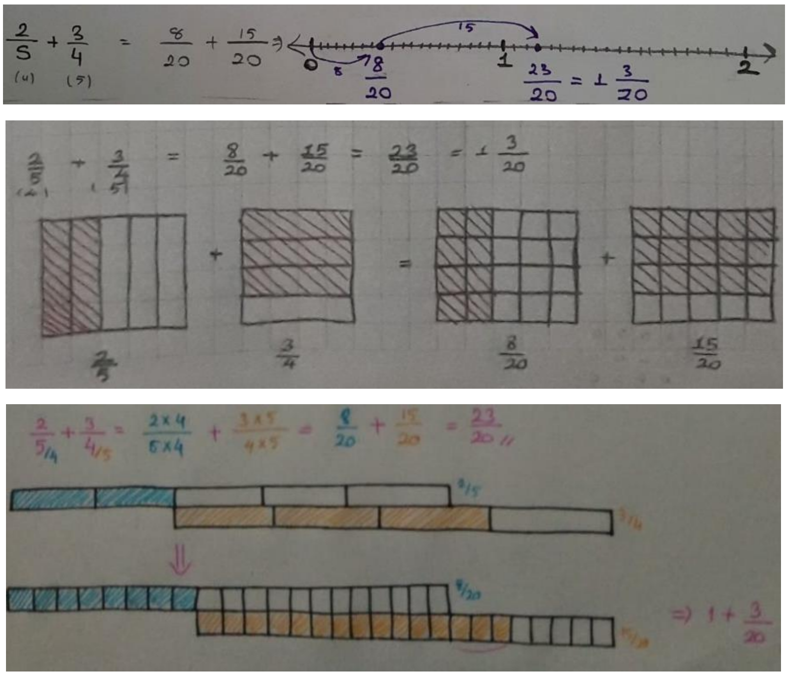


Figure 11. Models by P6, P42 and P51 concerning addition in rational numbers

Moving from the principles of teaching fractions, PSTs-M modelled the sample addition with rational numbers using the numerical axis, length and area models (Figure 10). One can also see that PSTs-M paid particular attention to the axiomatic structure in their definitions and explanations of the operations performed with rational numbers. During the later stages of the process, PSTs-M were given the operation $\frac{2}{5}\times\frac{3}{4}$ and asked to elaborate on the operation. For this operation, P1 indicated that "*it signifies finding 3/4 of 2/5*", explaining the meaning of the operation using the area model (Figure 12).

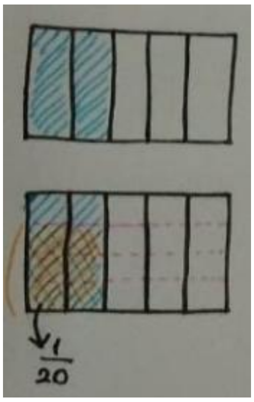


Figure 12. Modelling multiplication with rational numbers

As far as the division with rational numbers is concerned, the sample statement of $\frac{1}{4}\div\frac{5}{16}$ was given to PSTs-M and they were asked about the significance of this expression. PSTs-M replied to this question by stating that "*it denotes the question of how many* $\frac{5}{16}$ *are there in* $\frac{1}{4}$". As for P12 and P59, they made an operational definition based on multiplication by indicating that "*one solves [the equation] by considering the multiplicative inverse of the second number*". Some of the PSTs-M made explanations based on operation process from an example not based on the definition of the concept. Based on these definitions, one can argue that PSTs-M were able to define and explain division with rational numbers in a conceptual and operational manner during the CTD stage.

3.5. Definitions by PSTs-M for the concept of irrational numbers during the pre-CTD/CTD phases

The definitions given by PSTs-M during the pre-CTD stage concerning the concept of irrational numbers and the properties of operations performed with irrational numbers were examined; Table 11 shows these definitions.

Table 11.
Definitions by PSTs-M for the Concept of Irrational Numbers During the pre-CTD Phase

Concept	Definition	f	%
Irrational number	Numbers that cannot be written as a/b	29	43
	Non-rational numbers	39	57
2,474747…	Is an irrational number	10	15
	Is not an irrational number	58	85
182,02002000200002…	Is an irrational number	68	100

During the pre-CTD stage, most PSTs-M defined irrational numbers as "*non-rational numbers that cannot be shown as a/b*". Upon being presented with decimal representations repeating regularly or irregularly, PSTs did not agree on the question if the number is rational or irrational, as it was the case for the example of 2,474747…. The sample explanations provided by pre-service teachers are given below.

> ***P4:*** *It is an irrational number because it is not equal to a fixed value.*
> ***P12:*** *The decimal part of the number continues infinitely, and we cannot express is in the a/b format; therefore, it is irrational.*
> ***P36:*** *It is an irrational number because it does not denote a fixed value. The use of … indicates that the decimal goes on. The decimal part of 47 is continuously repeated after 2,47.*
> ***P38:*** *It is not irrational because it goes on based on a certain rule after the decimal.*
> ***P43:*** *It is a rational number. We can put it in the rational format as it has a repeating decimal.*

It is seen that PSTs-M do not have a correct understanding of the concept in their explanations that 2,474747… is an irrational number. Those claiming that the number does not denote a fixed value and that it cannot be shown as a/b indicates erroneous understandings concerning the topics of terminating and repeating decimals. To reveal the mis-conceptualization among PSTs-M in a more clear-cut manner, they were given the number 182,02002000200002… and asked whether it was an irrational number. All PSTs-M described this number as irrational. The sample explanations provided by PSTs-M are given below.

> ***P6:*** *It is irrational because the repetition is not regular. There is an approximate value; it is an irrational number. It does not indicate a fixed value on the numerical axis.*
> ***P36:*** *It is irrational, the decimal is not known clearly; there is no pattern [in the repetition].*
> ***P51:*** *We cannot put it in a rational format as the decimal does not repeat in a regular pattern; therefore, it is irrational.*

One can see that there are certain mis-conceptualization among PSTs-M concerning irrational numbers suggesting that "*[the number] does not denote a fixed value on the numerical line*". After these questions, PSTs-M were asked to show the number $\sqrt{3}$ on the numerical axis; Figure 13 shows the sample drawings created to answer this question.

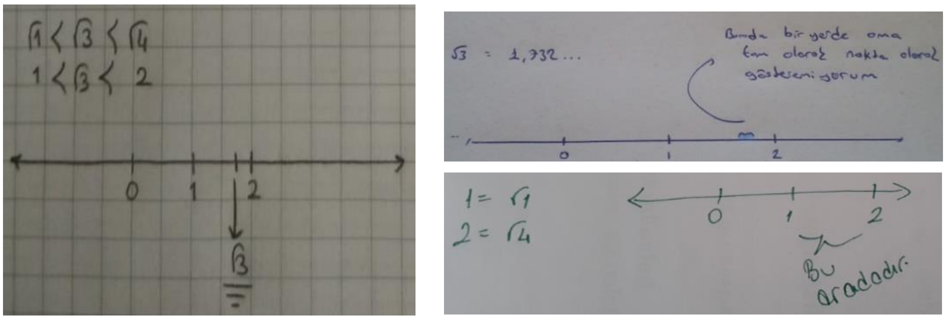


Figure 13. Indications of the positions of irrational numbers on the numerical axis

Taking the numbers $\sqrt{1}$ and $\sqrt{4}$ as reference points, P4, P15 and P50 showed the irrational number $\sqrt{3}$ to be somewhere between these two points, more specifically between 1 and 2. The sample drawings and explanations made by PSTs-M for finding the position of irrational numbers on the numerical axis demonstrate that they represent the point corresponding to the approximate value and make definitions intuitively. Table 12 shows the findings on the definitions by PSTs-M of irrational numbers as well as the operations they performed with irrational numbers.

Table 12.
Definitions by PSTs-M for the Concept of Irrational Numbers During the CTD Phase

Concept	Definition	f	%
Irrational number	Non-rational numbers	50	73
	Irregular decimal numbers	37	54
$\sqrt{3}+\sqrt{2}$	Is an irrational number	68	100

The definitions made by PSTs-M during the CTD stage for irrational numbers include statements such as "*non-rational numbers*" and "*irregular decimal numbers*". Some sample definitions by PSTs-M include expressions like "*the gap in the real numbers set left by rational numbers is filled by irrational numbers*" and "*the numbers that we cannot express in the rational format of a/b are irrational numbers*". On the basis of these definitions, one might say that PSTs-M define irrational numbers in an equivalent way suitable to conceptual hierarchy and satisfying required and sufficient conditions in the CTD stage.
PSTs-M were observed to argue that the expression "$\sqrt{3}+\sqrt{2}$" is "*an irrational number*". Based on this, they were also asked why they considered the expression as an irrational number, revealing that they solve the question based on proofs (Figure 14).

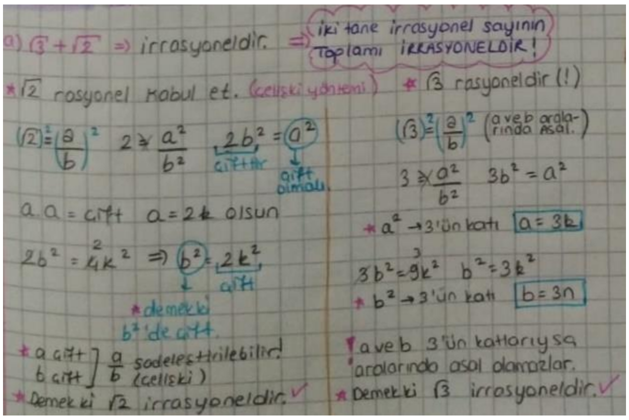


Figure 14. Sample justification for the irrational number statement

The irrationality of "$\sqrt{3} + \sqrt{2}$" was proven by P5 by means of contradiction. Based on this, it can be seen that many PSTs-M make definitions for the concept of irrational numbers and justifications in light of rational numbers. In this context, one might also argue that the definitions by PSTs-M affect the way they justify their solutions, and they make definitions in accordance with the axiomatic structure. Additionally, an in-class discussion session was initiated on the topic of whether irrational numbers correspond to a specific point on the numerical axis and the question of how to identify this point. Figure 15 features the sample drawings made by P4 and P7 to find the position of irrational numbers on the numerical axis.

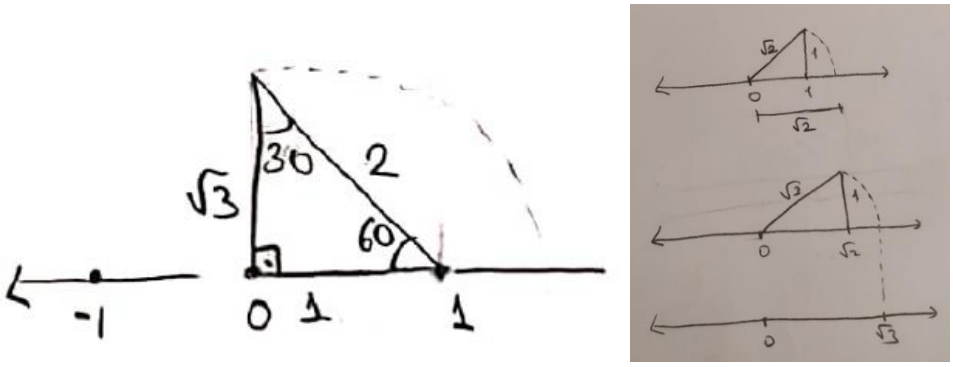


Figure 15. Sample representations of irrational numbers on the numerical axis during the CTD stage

> ***R:*** *Can we locate the irrational number $\sqrt{3}$ on the numerical axis? Or does it have a fixed place since it is irrational?*
> ***P47****: Irrational numbers fill the gaps left by rational numbers; therefore, they must correspond to certain points on the numerical axis.*
> ***P4:*** *Let us consider the triangle with two 1-unit sides and a hypotenuse of $\sqrt{2}$ units. For the number $\sqrt{3}$, if we draw a right-angled triangle with one side of 1 unit and a hypotenuse of 2 units, the length of the other side would be $\sqrt{3}$ units. If we draw this length on the numerical axis using a calliper, we can find the corresponding point.*
> ***P7:*** *Another solution would be drawing a unit circle, showing the point corresponding to the hypotenuse of $\sqrt{2}$ units with the calliper, and drawing a right-angled triangle with legs of 1 unit and $\sqrt{2}$ units with a hypotenuse of $\sqrt{3}$ units. Then, we can find the point corresponding to $\sqrt{3}$ on the numerical axis with the calliper.*

It can be seen that during the CTD stage, PSTs-M were able to identify the geometrical position of irrational numbers on the numerical axis through drawings and to explain the correspondence to a point based on definitions. Furthermore, they also explained the value corresponding to $\sqrt{3}$ in light of the operations made to find the number $\sqrt{2}$. This indicates a hierarchical approach. Additionally, this shows that PSTs-M used definitions in their explanations and justifications. Therefore, the initial mis-conceptualization among PSTs-M during the pre-CTD stage arguing that irrational numbers do not correspond to a particular point on the numerical axis seem to have changed.

3.6. Definitions by PSTs-M for the concept of real numbers during the pre-CTD/CTD phases

Table 13 shows the definitions by PSTs-M for the concept of real numbers during the pre-CTD stage analysed for the study.

Table 13.
Definitions by PSTs-M for the Concept of Real Numbers During the pre-CTD Phase

Concept	Definition	f	%
Real number	Union of natural, rational and irrational numbers	8	12
	Union of natural numbers, integers, rational numbers, irrational numbers and complex numbers	3	4
	Union of natural numbers, counting numbers, integers, rational numbers and irrational numbers	6	9
	Union of natural numbers, integers, rational numbers and irrational numbers	51	75

The definitions of real numbers by PSTs-M seem to include natural numbers, counting numbers, integers, rational numbers, irrational numbers and complex numbers. They expressed their definitions on the basis of the union of number sets. In these unions, counting numbers and complex numbers were also included. From these definitions, one can see that some PSTs-M were not able to define the number sets constituting real numbers and had certain mis-conceptualization. However, most PSTs-M could elaborate on real numbers correctly by stating that "*it is the union of natural numbers, integers, rational numbers and irrational numbers*". Therefore, the majority of PSTs-M seem to be able to identify the number sets included in real numbers accurately. Table 14 shows the definitions by PSTs-M for the concept of real numbers during the CTD stage analysed for the study.

Table 14.
Definitions by PSTs-M for the Concept of Real Numbers During the CTD Phase

Concept	Definition	f	%
Real number	All points on the numerical axis	14	20
	Union of natural numbers, integers, rational numbers and irrational numbers	68	100
	Real numbers	58	85

The definitions of real numbers provided by PSTs-M in the CTD stage seem to involve explanations based on number sets and the numerical axis. Based on these definitions, one might see that PSTs-M fulfil the required and sufficient conditions and satisfy the criteria of equivalence and axiomatisation in their definitions. Furthermore, they also made accurate explanations while expressing the number sets constituting real numbers. PSTs-M also seem to include explanations on real numbers in terms of number sets constituting real numbers prior to the instruction of the concept of real numbers. For example, the definitions made by PSTs-M for the concept of sets included statements such as "*...the set of real numbers is also infinite, but they have different cardinalities*" and "*in the real numbers set, the gap left by rational numbers is filled by irrational numbers*". Based on these explanations, one might reach the conclusion that PSTs-M have an advanced comprehension of real numbers in the CTD stage.

4. RESULTS, DISCUSSION AND RECOMMENDATIONS

This section deals with the results concerning the way PSTs-M define and conceptualise mathematical concepts in the pre-CTD and CTD stages, discusses these results within the context of the existing body of literature, and, ultimately, puts forward suggestions based on the study outcomes. The study concentrated on sets; number sets containing natural numbers, integers, rational numbers, irrational numbers, and real numbers; and concepts related to these topics. Within this scope, the definitions and conceptualisations made by PSTs-M concerning the concepts and operations covered in these topics were obtained; the study outcomes were then outlined accordingly.

During the pre-CTD stage, PSTs-M defined sets intuitively and did not have definitive knowledge of the concepts of equal and equivalent sets, describing the universal set as a set with many elements. Based on these findings, it was concluded that PSTs-M were not able to express the mathematical meanings of the concepts of sets, equivalence, equality, and universal sets clearly in the pre-CTD stage. In this context, their explanations for the concepts in question seemed to be based on concept images rather than definitions. As for the CTD stage, PSTs-M were seen to provide definitions for the concept of sets in line with the naive and axiomatic set theories. Based on the explanations provided, the definitions made for universal sets and finite/infinite sets were fit for the axiomatic structure and equivalent in nature. In the CTD stage, particularly while trying to convince the researcher and each other, they demonstrated proofs and justifications based on the definition of sets. The findings obtained in the CTD stage led to the conclusion that definitions affect the way PSTs-M define and conceptualise the concept of sets.

As far as the natural number definitions given by PSTs-M during the pre-CTD stage are concerned, they seemed to be based on integers, which is not suitable for conceptual hierarchy. It was also observed that PSTs-M did not know about the "closure" property in operations performed with natural numbers or the circumstances leading to this feature. Dickerson and Pitnam (2016) asked them to define the concepts of even number, prime number, square, parallel line, tangent and trapezoid in their study with 10 undergraduate mathematics students. Similar to this study, it was concluded that mathematics students were insufficient in writing the definition of the related concept in their minds and they had a limited understanding. During the CTD stage, however, natural numbers were defined on the basis of the Peano axioms by PSTs-M. Therefore, the researchers reached the conclusion in this process that PSTs-M provided axiomatic and equivalent definitions fulfilling the required and sufficient conditions for the concept in question. As for the operations made using natural numbers, modelling was the preferred course of action. While addition and subtraction were explained using counting pieces and the numerical axis, multiplication was described using the area model. The operations based on these points were explained by PSTs-M in a conceptual manner. In light of these results, it was concluded that definitions affected the way PSTs-M conceptualise the concepts covered in the course. The lack of the closure property in subtraction and multiplication were expressed by PSTs-M in line with the principle of conceptual hierarchy based on the concept of identity elements while defining operations, leading to a generalisation arguing that "*every number set is defined based on natural numbers*".

PSTs-M defined the concept of integers intuitively in the pre-CTD stage. The operations made with integers were defined by PSTs-M with the symbols (+) and (-) with an operational approach without indicating the mathematical meaning. In the CTD stage, however, the concept of integers was defined on the basis of natural numbers, which is suitable in terms of conceptual hierarchy. Furthermore, alternative definitions were also provided using mathematical symbols. These definitions made by PSTs-M for the concept of integers were in line with the principle of conceptual hierarchy, included the required and sufficient conditions, and were equivalent and economical. The notations of (+) and (-) used in operations with integers were defined as operations and directions; counting pieces and numerical axis modelling were used to explain these definitions. Consequently, it was concluded that definitions influenced the way integers and the operations made with integers are conceptualised.

For the definition of rational numbers in the pre-CTD stage, the approach followed by pre-service teachers were rather intuitive, starting from fractions and expressing them as a/b. The operations made with rational numbers were expressed in either operational or algorithmic terms. Similarly, Depaepe et al. (2018) states that pre-service mathematics teachers have difficulties in defining rational numbers and decimal notation and mathematical depth of operations, and they lack knowledge. Weller, Arnon, and Dubinsky (2009) implemented an experimental teaching model in their work with pre-service teachers. They concluded that the pre-service teachers in the control group where traditional teaching was applied had less progress in procedural and conceptual understanding of rational numbers, fractions, and decimal notation. In the CTD stage, however, PSTs-M defined rational numbers in line with the principle of conceptual hierarchy based on the lack of closure property under multiplication with integers and natural numbers. Furthermore, they made use of mathematical symbols in their definitions, benefiting from multiple representations. This shows that CTD influences the use of mathematical language by PSTs-M. Based on these findings, the researchers reached the conclusion that the definitions by PSTs-M concerning rational numbers in the CTD stage complied with conceptual hierarchy, indicated the required and sufficient conditions, and satisfied the criteria of equivalence, being economical, and elegance. The definitions and explanations made by PSTs-M adopted a conceptual approach when it came to the operations performed with rational numbers. They provided examples for addition and subtraction with rational numbers using the area model, length model, and the numerical axis. The area model was also preferred for the explanation of multiplication while separate examples were given for division in a conceptual manner. It is concluded that the concept definitions of PSTs-M were effectiveness in the systematization and axiomatic structure of the proofs.

In the pre-CTD stage, PSTs-M defined the concept of irrational numbers based on rational numbers, which corresponds to the principle of conceptual hierarchy. However, upon being given decimals repeating either regularly or irregularly, when PSTs-M were asked whether these decimal numbers are irrational, they were not able to interpret and explain these examples correctly. Therefore, their irrational number definitions seemed to be intuitive. Similarly, Guven, Cekmez and Karatas (2011) revealed in their study that pre-service teachers define irrational numbers and represent them on the number line intuitively. PSTs-M claimed that irrational numbers cannot be placed on an exact point on the numerical axis as they do not correspond to an exact value and that they can be represented as approximate values within a range. These explanations indicated mis-conceptualization among PSTs-M concerning irrational numbers. In the CTD stage, however, they defined irrational numbers based on rational numbers and decimals that had been defined beforehand. This satisfies the criterion of conceptual hierarchy. As for the representation of irrational numbers on the numerical axis, PSTs-M seemed to adopt an inductive approach, indicating geometric positions. In the CTD stage, the irrational of these numbers was questioned; PSTs-M indicated mathematical proofs in their explanations for this question. The definition of irrational numbers was employed in the proofs and explanations provided. As far as the results for this concept as well as other mathematical concepts considered in the study are concerned, one can reach the conclusion that definitions support the way one provides mathematical proofs.

The definitions by PSTs-M for the concept of real numbers reveal that they list the wrong number sets in the pre-CTD stage. In particular, the inclusion of counting numbers and complex numbers reveal the mis-conceptualization regarding real numbers. Furthermore, the majority of PSTs-M indicated the number sets constituting real numbers accurately. The detailed analysis of the number sets constituting real numbers in the study led to the conclusion that PSTs-M knew about the number sets forming the set of real numbers; but they still lacked certain capacities in defining and conceptualising these number sets in the pre-CTD stage. The definitions by PSTs-M in the CTD stage, however, were based on the real number sets defined previously as well as on the concept of numerical axis studied beforehand. In light of these definitions, one might argue that the mis-conceptualization among PSTs-M on the subject of real numbers was eliminated in the CTD stage, developing their conceptual understanding of real numbers.

Although it is expected that the pre-service teachers who encounter different definitions, shapes and examples at different education levels are expected to create more advanced explanations and informal definitions (Karakuş, 2018), in this study, as a result of the examination of the definitions made by PSTs-M on the subject in the CTD, it was seen that the participants were unable to make correct definitions or define the concepts concerned at all due to their lack of sufficient knowledge. Similarly, in the study carried out by Ünlü (2021), pre-service teachers' definitions of circle, circular region, and sphere concepts were examined, and the study revealed that pre-service teachers had difficulties in defining the concepts. The study by Vinner (1977) found that students do not have an exact knowledge of axioms and propositions due to the lack of a complete understanding of the nature of definitions, incorrectly believing that the proven proposition is the definition. Therefore, the fact that PSTs-M were familiar with the nature of mathematical concepts and able to make definitions shows that mathematical concepts allow them to understand the process of conceptualisation. Dickerson and Pitman (2016) stated that students were unable to write definitions while expressing their opinions on concepts. However, owing to the presentation of definitive criteria and setting up discussion sessions among pre-service teachers in the CTD stage allowed them to come up with suitable definitions in the consideration of definitive criteria. The present study is based on the obligatory and optional criteria for definitions proposed by Zaslavsky and Shir (2005). Based on this, it can be seen that PSTs-M fulfil the obligatory criteria of axiomatisation, equivalence and inclusion of required and sufficient conditions while defining the concepts of sets, natural numbers, integers, rational numbers, irrational numbers and real numbers in the CTD stage. Furthermore, as far as the definitions for integers and rational numbers are concerned, they also satisfy the optional criteria of elegance and being economical in addition to the obligatory ones. In line with these results, it can be concluded that CTD is effectiveness in the definition of number sets by PSTs-M. Similarly, Zaslavsky and Shir (2005) conducted a group-oriented study, concentrating on the definitions made by high-school students, and indicated that the discussion environments provided affected the way students made definitions.

Certain mis-conceptualization was observed among PSTs-M in the pre-CTD stage, concerning some of the concepts addressed in the study. Defining the universal set as a set with many elements, considering rational numbers as fractions, expressing fractions with numerical values, claiming that irrational numbers do not correspond to a certain point on the numerical axis as they have repeating decimals are some examples of the mis-conceptualization in the definitions made by PSTs-M in the pre-CTD stage. With the exposure of PSTs-M to concept definitions and applications in the CTD stage, these mis-conceptualization were eliminated. Dealing with examples presenting contradictory cases for learners, leads to a cognitive conflict, and the resolution of these contradictions is possible through the change of knowledge (Lakatos, 1976; Peled & Zaslavsky, 1997; Zaslavsky & Shir, 2005). Implemented based on this argument, CTD was found to be effectiveness in the elimination of mis-conceptualization among PSTs-M. PSTs-M were generally observed to make intuitive definitions and base their explanations on concept images rather than mathematical meaning while defining mathematical concepts in the pre-CTD stage. This finding is similar to the outcomes of the study by Edwards and Ward (2004) conducted with undergraduate mathematics students. Edwards and Ward (2004) stated in their study that students are prone to think that the mathematical definition of the concept in question might be wrong when the indicated definition contradicts their own concept images. In this respect, it is important for students to abandon their personal conceptions and gain experience in terms of adopting mathematical definitions. Therefore, PSTs-M must design suitable mathematical experiences in accordance with didactic activities to help pre-service teachers understand the nature and roles of mathematical definitions.

The present study concludes that CTD affects the way PSTs-M define mathematical concepts in parallel to the fact that it also influences their conceptualisations and proofs. The ability of PSTs-M to accurately define mathematical concepts seem to influence the way they prove their operations and actions. Similarly, Vinner (1977) stated that students do not know how to use definitions for proofs, arguing that the reason behind this is their lack of knowledge on the meaning of definitions. CTD was observed to affect the way PSTs-M make definitions and use mathematical notations and verbal expressions, contributing to their mathematical language and communication skills. Similarly, Moore (1994) indicated that definitions have a positive impact on the mathematical language and communication skills of students. On the basis of this idea, the study considers the teaching of definitions ensures the meaningful instruction of mathematical concepts. In a study conducted with university-level students of mathematics, Dickerson and Pitman (2016) stated that even though the students in question receive advanced mathematics education, they were unable to define basic concepts like integers, prime numbers, even numbers and real numbers, not knowing their conceptual significances. CTD is proven to be effectiveness in revealing and eliminating mis-conceptualization and false concept images among PSTs-M. This is because definitions play a significant role in revealing mis-conceptualization, weak concept images and inaccurate definitions (Dickerson & Pitman, 2016). During the CTD stage, PSTs-M justified their explanations and definitions based on axioms and definitions. The analysis of these processes led to the conclusion that definitions affect the way PSTs-M demonstrate proofs, provide justifications, establish cause-effect relationships, and make generalisations. The study conducted by Zaslavsky and Shir (2005) with 12th-grade students similarly revealed that definition-based activities contribute to the proving skills of students, allowing students to make explanations and provide reasoning based on definitions.

In line with the outcomes obtained from the present study, the following suggestions might be made for future studies:

1. Definitions might be included and prioritised for the development of area-specific knowledge of pre-service mathematics teachers concerning mathematical concepts during their undergraduate studies. The inclusion of definitions in their subject-specific courses on mathematics may allow pre-service teachers to make sense of mathematical concepts and definitions at the conceptual level.

2. The study was conducted during the course entitled "Fundamentals of Mathematics I" and was limited to the scope of the number sets dealt with throughout the course. Definition-based teaching for the instruction of different courses and concepts may allow future researchers to focus on different concepts. This would allow for the creation of a wider working network for the effectiveness of definitions for mathematical concepts.
3. The present study was conducted with pre-service middle-school mathematics teachers and identified the effectiveness of teaching through definitions on the definition and conceptualisation capacity of pre-service teachers. A similar syllabus may be applied through in-service training programmes, enabling the organisation of activities in terms of definition and conceptualisation capacities of actively working teachers as well. Therefore, one might be able to identify the post-service effectiveness of definitions for teachers as well as their pre-service effectiveness within the framework of teacher training.

Research and Publication Ethics Statement

The present study devotes particular concern to the research publication and ethics declaration. Ethical permissions and participant consent were sought, and participant privacy was protected during the course of the study. This study was found ethically appropriate by the decision of Yozgat Bozok University Institutional Review Board dated 20.01.2021 and numbered E-95799348-050.01.04-3198.

Contribution Rates of Authors to the Article

Each author contributed equally to the drafting of the present article.

Statement of Interest

The authors declare that they have no known conflict of interest.

5. REFERENCES

Ball, D. L, & Bass, H. (2000). Making believe: The collective construction of public mathematical knowledge in the elementary classroom. In D. Phillips (Ed.), *Yearbook of the national society for the study of education, constructivism in education.* Chicago, IL: University of Chicago Press.

Ball, D. L. (1990). The mathematical understandings that prospective teachers bring to teacher education. *The Elementary School Journal, 90*(4)*,* 449-466.

Ball, D. L. (2003). *What mathematics knowledge is needed for teaching mathematics*? Paper presented at *the US Department of Education, Secretary's Mathematics Summit*, Washington, DC.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special. *Journal of Teacher Education*, *59*(5), 389-407. doi: 10.1177/0022487108324554

Borasi, R. (1992). *Learning mathematics through inquiry.* Portsmouth, NH: Heinemann Educational Books.

Borko, H., & Putnam, R. (1996). Learning to teach. In R. Calfee & D. Berliner (Eds.), *Handbook of educational psychology* (pp. 673-725). New York: Macmillan.

Counchil of Higher Education. (2018). *Elementary mathematics education undergraduate program*. Retrieved from https://www.yok.gov.tr/Documents/Kurumsal/egitim_ogretim_dairesi/Yeni-Ogretmen-Yetistirme-Lisans-Programlari/Ilkogretim_Matematik_Lisans_Programi.pdf

de Villiers, M. (1998). To teach definitions in geometry or to teach to define? In A. Olivier & K. Newstead (Eds.), *Proceedings of the 22nd Conference of the International Group for the Psychology of Mathematics Education* (pp. 248-255). Stellenbosch, South Africa: University of Stellenbosch.

Depaepe, F., Van Roy, P., Torbeyns, J., Kleickmann, T., Van Dooren, W., & Verschaffel, L. (2018). Stimulating pre-service teachers' content and pedagogical content knowledge on rational numbers. *Educational Studies in Mathematics, 99*(2), 197-216. doi: 10.1007/s10649-018-9822-7

Dickerson, D. S., & Pitman, D. J. (2016). An examination of college mathematics majors' understandings of their own written definitions. *The Journal of Mathematical Behavior*, *41*, 1-9. doi: 10.1016/j.jmathb.2015.11.001

Dickerson, D., & Pitman, D. (2012). Advanced college-level students' categorization and use of mathematical definitions. In Tso, T. Y. (Ed.). *Proceedings of the 36th Conference of the International Group for the Psychology of Mathematics Education* (pp. 187-193). Taipei, Taiwan: PME.

Edwards, B., & Ward, M. (2004). Surprises from mathematics education research: Student (mis)use of mathematical definitions. *The American Mathematical Monthly*, *111*(5), 411-424. doi: 10.1080/00029890.2004.11920092

Fischbein, E., Jehiam, R., & Cohen, D. (1995). The concept of irrational numbers in high-school students and prospective teachers. *Educational Studies in Mathematics, 29*(1), 29-44. doi: 10.1007/BF01273899

Fraenkel, J. R., & Wallen, N. E. (2009). *The nature of qualitative research. How to design and evaluate research in education*. Boston: McGraw-Hill.

Freudenthal, H. (1973). *Mathematics as an educational task.* Dordrecht, The Netherlands: D. Reidel.

Gilboa, N., Kidron,I., & Dreyfus, T. (2019). Constructing a mathematical definition: the case of the tangent. *International Journal of Mathematics Education in Science and Technology, 50*(3), 421-436. doi: 10.1080/0020739X.2018.1516824

Guven, B., Cekmez, E., & Karatas, I. (2011). Examining preservice elementary mathematics teachers' understandings about irrational numbers. *PRIMUS*, *21*(5), 401-416. doi: 10.1080/10511970903256928

Harel, G. (2008). DNR perspective on mathematics curriculum and instruction, Part II: With reference to teacher's knowledge base. *Zentralblatt fur Didaktik der Mathematik*, *40*, 893-907. doi: 10.1007/s11858-008-0146-4

Johnson, H. L., Blume, G. W., Shimizu, J. K., Graysay, D., & Konnova, S. (2014). A teacher's conception of definition and use of examples when doing and teaching mathematics. *Mathematical Thinking and Learning*, *16*(4), 285-311. doi: 10.1080/10986065.2014.953018

Karakuş, F. (2018). Investigation of primary pre-service teachers' concept images on cylinder and cone. Elementary Education Online, 17(2), 1033-1050. doi: 10.17051/ilkonline.2018.419352

Karatas, I., Guven, B., & Cekmez, E. (2011). A cross-age study of students' understanding of limit and continuity concepts. *Boletim de Educação Matemática, 24(*38), 245-264. doi: 10.1590/S0103-636X2013000400007

Khinchin, A. Y. (1968). *The teaching of mathematics*. London: The English Universities Press.

Lakatos, I. (1976). *Proofs and refutations*. Cambridge, UK: Cambridge.

Landau, S. I. (2001). *Dictionaries: The art and craft of lexicography* (2nd ed). Cambridge: Cambridge University Press.

Leikin, R., & Winicki-Landman, G. (2000). On equivalent and non-equivalent definitions II. *For the Learning of Mathematics*, *20*(2), 24-29.

Levenson, E. (2012). Teachers' knowledge of the nature of definitions: The case of the zero exponent. *Journal of Mathematical Behavior*, *31*(2), 209-219. doi: 10.1016/j.jmathb.2011.12.006

Liljedahl, P., Sinclair, N., & Zazkis, R. (2006). Number concepts with Number Worlds: thickening understandings. *International Journal of Mathematical Education in Science and Technology, 37*(3), 253-275. doi: 10.1080/00207390500285909

Mariotti, M. A., & Fischbein, E. (1997). Defining in classroom activities. *Educational Studies in Mathematics*, *34*, 219-248. doi: 10.1023/A:1002985109323

McCrory, R., & Stylianides, A.J. (2014). Reasoning-and-proving in mathematics textbooks for prospective elementary teachers. *International Journal of Educational Research, 64,* 119-131. doi: 10.1016/j.ijer.2013.09.003

Merriam, S. B. (1988). *Case study research in education: A qualitative approach.* San Francisco: Jossey-Bass.

Michener, E. R. (1978). Understanding understanding mathematics. *Cognitive Science, 2*(4), 361–383. doi: 10.1207/s15516709cog0204_3

Ministry of National Education. (2018). *Matematik dersi öğretim programı (İlkokul ve ortaokul 1, 2, 3, 4, 5, 6, 7 ve 8. sınıflar)* [Mathematics curriculum: Elementary and middle schools 3, 4, 5, 6, 7 and 8th grades)]. Ankara: Talim Terbiye Kurul Başkanlığı.

Moore, R. C. (1994). Making the transition to formal proof. *Educational Studies in Mathematics*, *27*, 249-266. doi: 10.1007/BF01273731

Peled, I., & Zaslavsky, 0. (1997). Counter-examples that (only) prove and counter-examples that (also) explain. *Focus on Learning Problems in Mathematics, 19*(3), 49-61.

Pimm, D. (1993). Just a matter of definition [Review of the book Learning mathematics through inquiry]. *Educational Studies in Mathematics*, *25*(3), 261-277.

Polya, G. (1957). *How to solve it: A new aspect of mathematical method.* Garden City, NY: Doubleday.

Robinson, R. (1962). *Definitions*. London: Oxford University Press.

Saxe, G., Gearhart, M., Diakow, R., Buchanan, N., Collett, J., Kang, B., Kirby, K., & Le, M. (2013). *Engagement in mathematical discussion: Linking practices and outcomes.* Presentation at Research Pre-Session of National Council of Teachers of Mathematics, Denver, CO.

Seaman, C. E., & Szydlik, J. E. (2007). Mathematical sophistication among preservice elementary teachers. *Journal of Mathematics Teacher Education*, *10*, 167-182. doi: 10.1007/s10857-007-9033-0

Sirotic, N., & Zazkis, R. (2007). Irrational numbers: the gap between formal and intuitive knowledge. *Educational Studies in Mathematics, 65*, 49–76. doi: 10.1007/s10649-006-9041-5

Smith III, J. P., Males, L. M, Dietiker, L. C., Lee, K., & Moiser, A. (2013). Curricula treatments of length measurement in the United States: Do they address know learning challenges? *Cognition and Instruction*, *32*(4), 388-433. doi: 10.1080/07370008.2013.828728

Stake, R. E. (1995). *The art of case study research.* London: Sage.

Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics*, *12*, 151-169. doi: 10.1007/BF00305619

Tirosh, D. (2000). Enhancing prospective teachers' knowledge of children's conceptions: The case of division of fractions. *Journal for Research in Mathematics Education, 31*(1), 5–25. doi: 10.2307/749817

Tirosh, D., & Even, R. (1997). To define or not to define: The case of $(-8)^{1/3}$. *Educational Studies in Mathematics*, *33*(3), 321-330. doi: 10.1023/A:1002916606955

Toluk-Uçar, Z. (2016). The role of representations in middle school preservice teachers' conceptions of real numbers. *Kastamonu Eğitim Dergisi, 24*(3), 1149-1164.

Usiskin, Z., Hirschhorn, D., Coxford, A., Highstone, V., Lewellen, H., Oppong, N., Dibianca, R., & Maeir, M. (1997). *Geometry: The University of Chicago school mathematics project.* Glenview, IL: Scott, Foresman and Company.

Ünlü, M. (2021). Investigation of preservice mathematics teachers' concept definitions of circle, circular region, and sphere. *International Journal of Mathematical Education in Science and Technology,* 1-28. doi: 10.1080/0020739X.2020.1847334

Van Dormolen, J., & Zaslavsky, O. (2003). The many facets of a definition: The case of periodicity. *Journal of Mathematical Behavior, 22*(1), 91-196. doi: 10.1016/S0732-3123(03)00006-3

Vinner, S. (1977). The concept of exponentiation at the undergraduate level and the definitional approach. *Educational Studies in Mathematics, 8*, 151-169. doi: 10.1007/BF00302501

Vinner, S. (1991). The role of definitions in the teaching and learning of mathematics. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 65-81)*.* Dordrecht, Netherlands: Kluwer Academic Publishers.

Weber, K. (2002). Beyond proving and explaining: Proofs that justify the use of definitions and axiomatic structures and proofs that illustrates technique. *For the Learning of Mathematics*, *22*(3), 14-22.

Weller, K., Arnon, I., & Dubinsky, E. (2009). Preservice teachers' understanding of the relation between a fraction or integer and its decimal expansion. *Canadian Journal of Science, Mathematics and Technology Education, 9*(1), 5-28. doi: 10.1080/14926150902817381

Wilson, P. S. (1990). Inconsistent ideas related to definitions and examples. *Focus on Learning Problems in Mathematics, 12*(3-4), 31-47.

Winicki-Landman, G., & Leikin, R. (2000). On equivalent and non-equivalent definitions I. *For the Learning of Mathematics, 20*(1), 17-21.

Yıldırım, A., & Şimşek, H. (2016). *Sosyal bilimlerde nitel araştırma yöntemleri. (9th ed.)* Ankara: Seçkin Yayınevi.

Zaslavsky, O., & Shir, K. (2005). Students' conceptions of a mathematical definition. *Journal for Research in Mathematics Education*, *36*(4), 317-346. doi: 10.2307/30035043

Zazkis, R., & Leikin, R. (2008). Exemplifying definitions: a case of a square. *Educational Studies in Mathematics*, *69*, 131-148. doi: 10.1007/s10649-008-9131-7