Hacettepe University Journal of Education (H. Ü. Eğitim Fakültesi Dergisi) 37*(3), 1154-1175* [2022]

Hacettepe University Journal of Education

Hacettepe Üniversitesi Eğitim Fakültesi Dergisi

e-ISSN: 2536-4758

The Analysis of the Algebraic Proving Process Based on Habermas' Construct of Rationality*

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Article Information

Received: 17.09.2020

Accepted: 19.05.2021

Online First: 07.06.2021

Published: 31.07.2022

ABSTRACT

Proving is considered to be one of the most important activities in mathematics. Some of the studies on mathematical proof provide various useful tools, which enable us to analyze the proving processes of students. Recent studies have shown that the proving process of students needs to be evaluated comprehensively and coherently considering the proving strategies students use and the mathematics level of the community students are in and they communicate with. To this end, Habermas' construct of rationality has been used by the researchers to analyze some mathematical activities, such as problem-solving, proving, and modeling. Habermas' construct of rationality is composed of three integrated components which are epistemic, teleological, and communicative rationality. This study is a qualitative case study, which aims to analyze the proving processes of university students in the field of algebra within the context of these rationality components. The results of the study revealed that the algebraic proving process of the students has substantially been affected by the interaction between the rationality components. Furthermore, based on the needs that arose during the analyses, it is recommended to add new sub-components to the modeling requirements of the epistemic rationality and the communicative rationality components of Habermas' construct of rationality.

Keywords: Algebraic proof, proving process, Habermas' construct of rationality, algebra

doi: 10.16986/HUJE.2021069742 Article Type: Research Article

Citation Information: Urhan, S., & Bülbül, A. (2022). The analysis of the algebraic proving process based on Habermas' Construct of Rationality. *Hacettepe University Journal of Education, 37*(3), 1154-1175. doi: 10.16986/HUJE.2021069742

1. INTRODUCTION

Proving is referred to as the core of mathematical activities (Nardi & Knuth, 2017; Stylianides, Bieda & Morselli, 2016; Stylianides, Stylianides & Weber, 2017). It involves logical, conceptual, social, and problem-solving dimensions to demonstrate the accuracy of expressions and to ensure precision in mathematics (Mejia-Ramos & Inglis, 2008). Many researchers in the field of mathematics education consider proving to be a learning tool based on the fact that it involves confirmation of the assertions and understanding the reason why they are true while developing or refining one's representations and intuitions about mathematical concepts (Hanna, 1990; Pinto & Tall, 1999; Rodd, 2000).

Proving can be investigated from different perspectives of the field of mathematics education. However, regardless of which perspective is adopted, the complexity of proving cannot be thoroughly explained (Boero, 2006; Morselli & Boero, 2009, 2011). Mathematical problem-solving research (Schoenfeld, 1992) has identified and described the student performance in problem solving activities. However, it has not yet addressed the cultural aspect and value of student performance considering different epistemological and cultural constraints (Boero, 2006). The theoretical construct of "theorem" as "statement, proof, and reference theory" (Mariotti, Bartolini Bussi, Boero, Ferri, & Garuti, 1997) may enable us to analyse and evaluate the proving process of students. However, the cognitive aspect of the process can not be evaluated and teachers' evaluation criteria cannot easily been explained or questioned within this framework (Boero, 2006). Duval's (2007) framework regarding the cognitive distance between argumentation and mathematical proof can be utilized to estimate the distance between the argumentative performance and the proof as formal derivation. However, this framework cannot account for the quality of the problem-solving strategy and teachers' evaluations (Morselli & Boero, 2011). Alternatively, Harel's proof schemes (Harel & Sowder, 1998) can

[*] This article is derived from a part of the PhD dissertation of the first author titled "Analysis of Proof Process Based on Habermas Rational Behaviour Construct", conducted under the supervision of the second author. The ethical approval for the research has been obtained from the Ethics Commission of Hacettepe University (decision number: 35853172/433-354) on January 17, 2017.

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be used to describe and evaluate the student performance in proving. However, these schemes cannot address the distance between the student's proving performance and the requirements of a proof text considering the teachers' evaluation criteria (Morselli & Boero, 2009).

Many research studies have so far investigated the challenges inherent in teaching and learning proof. These studies have offerred valuable insight into these challenges; however, students still have difficulties in learning to prove, and teachers still find it challenging to teach proof (Nardi & Knuth, 2017). As Stylianides et al. (2017) note, research has not provided teachers, and teacher educators with sufficient support as to how they might address the problems of practice in proving. According to the researchers (Boero, 2006; Morselli & Boero, 2009), there is a need for a new framework to deal with the epistemological issues inherent in the analysis of student products, evaluations of the teachers on these products, and to reveal the quality of the proving strategies used by the students regarding their communication within their community of practice. To this end, in recent years, Habermas' construct of rationality has been used as a tool to analyze the proving process of students considering the epistemic validity and the problem-solving nature of the process, the conformity between the communication methods and the communication rules of the field, and the understandability of the communication by others (Boero, 2006, 2017; Boero, Douek, Morselli, & Pedemonte, 2010; Boero, Guala, & Morselli, 2013; Boero & Morselli, 2009; Boero & Planas, 2014; Conner, 2018; Mariotti, Durand-Guerrier, & Stylianides, 2018; Morselli & Boero, 2009, 2011).

As Habermas (2003) puts forward, a person is considered to be acting rationally when s/he acts according to his/her own views during an activity, conforming to the criteria and the communication rules of the field to achieve his/her goal. Habermas' construct of rationality is composed of three interrelated components, which are knowledge at play (epistemic rationality), action and its goals (teleological rationality), and communication and related choices (communicative rationality). Epistemic rationality is concerned with the validity check of statements and inferences, combining the statements together consciously within a shared knowledge system or theory. Teleological rationality is about whether the activity is intentional and whether the person chooses and uses the convenient tools intentionally in line with his/her aim when performing the activity. The third component, communicative rationality, is related to the communication between the members of a community of practice. It requires choosing and using the means of communication consciously so that the aim of communication could be achieved.

Habermas' construct of rationality was adapted to the use of algebraic language in modeling and proving in mathematics education by Morselli and Boero (2009) and improved by the same researchers by adding sub-components under the epistemic rationality in their further study (Boero & Morselli, 2009). According to the adapted version of Habermas' construct of rationality, epistemic rationality is concerned with the validation of the statements, in terms of their correctness, rules of inference, and exact premises (Morselli & Boero, 2009). It involves modeling requirements (MR) and systemic requirements (SR) (Boero & Morselli, 2009). Modeling requirements refer to the coherence between the algebraic model and the modeled situation presented in the problem or the statement. Systemic requirements are related to the use and implementation of algebraic language and methods. To be more specific, systemic requirements involve manipulating the rules of the signs system such as the algebraic language, and implementing the methods to solve equations and inequalities accurately (Boero & Morselli, 2009). Teleological rationality involves choosing and finalizing algebraic formalizations, transformations and interpretations intentionally in order to serve the aims of an activity (Morselli & Boero, 2009). This rationality involves the management of the interaction between the author and the reader accurately and intentionally, when an author writes, transforms and interprets an algebraic expression in line with the aims of the activity (Boero & Morselli, 2009). Communicative rationality refers to the production of processes conforming to the standards of mathematics and its communication rules (Morselli & Boero, 2009). To meet the requirements of communicative rationality, social norms and the criteria regarding standard notations of mathematics should be followed in order to read and manipulate the algebraic expressions easily (Boero & Morselli, 2009).

The integration of Habermas' communicative perspective with the social interaction in mathematical activities such as proving may be difficult because such integration requires using concepts from different frameworks with different purposes (Boero & Planas, 2014). However, as Habermas' construct takes into consideration epistemic validity, strategic decisions taken during the process, and communication requirements; it may be useful to analyze the students' performances in mathematical activities (Mariotti et al., 2018). This construct allows establishing a connection between the individual and the community, considering the epistemic requirements of mathematical truth in a cultural context and the methods of discovering, ascertaining and communicating (Boero & Morselli, 2009; Boero & Planas, 2014; Morselli & Boero, 2011).

1.1. The Relationship between the Difficulties in Algebraic Proving and the Components of Habermas' Construct of Rationality

Swafford and Langrall (2000) defined algebraic reasoning as the ability to operate on an unknown quantity as if the quantity is known, in contrast to arithmetic reasoning, which involves operations with known quantities. According to Radford (2006), the major difference between arithmetic and algebraic thinking is that the former has numerical specificity, while the latter involves numerical uncertainty. Morselli and Boero (2011) maintained that the distance between arithmetic and algebra should be taken into consideration in order to establish the relationship between them within the proving process. The rules that still apply in arithmetic but no longer in algebra should be examined within the scope of epistemic rationality and teleological rationality, and the efficiency of algebra in solving more problems than arithmetic should be examined within the scope of teleological rationality.

Arcavi (2005) pointed to the importance of knowing how and when to use or not use algebra, choosing the best representation among different possibilities, and reading the symbols in algebraic proving. According to Morselli and Boero (2011), the first and second point that Arcavi mentioned are related to the requirements of teleological rationality, while the third one is about the requirements of communicative rationality according in the context of Habermas' construct of rationality.

Selden and Selden (2003) conducted a study within the scope of the Abstract Algebra course and determined certain misconceptions regarding the use of theorems, notations and symbols. According to Habermas' construct of rationality, the choice and use of theorems require students to choose the correct tool appropriate for their purpose and this can be examined within the scope of teleological rationality. On the other hand, the use of notations and symbols can be examined within the scope of communicative rationality.

Boero (2006) argued that students experience certain difficulties in proving due to the specific aspects of the use of algebraic language such as the correct interpretation of algebraic expressions in a given context (epistemic rationality), the goal-oriented nature of the choice of formalisms and of the direction of transformations (teleological rationality) as well as the restrictions resulting from communication rules (communicative rationality). VanSpronsen (2008) identified a number of difficulties students experienced during the proving process in algebra. The common difficulties were operational errors (systemic requirements of epistemic rationality), misuse of notations (communicative rationality) and misinterpretation of definitions (epistemic rationality), not knowing where to start and how to proceed (teleological rationality), focusing too much on a particular method of proving, and not being aware of the parts of the proof that would allow them to establish valid proofs (teleological rationality). In addition, VanSpronsen (2008) observed that students did not want to prove a proposition whose accuracy was clear (lack of communicative rationality and/or teleological rationality without sufficient epistemic rationality) and they preferred to end the proving process when they got stuck (lack of epistemic rationality and lack of teleological rationality) instead of trying out a new method.

Uğurel, Moralı, Yiğit Koyunkaya, and Karahan (2016) found that prospective secondary school mathematics teachers had difficulties in applying mathematical language and notations (communicative rationality), understanding the meaning of the given proposition (epistemic rationality), deciding where to start proving (teleological rationality), using examples efficiently (epistemic and teleological rationality), using appropriate and efficient methods to construct the proof (teleological rationality), and defining logical structures of the proposition to construct the proof (epistemic rationality).

It is deduced that the requirements of algebraic proving and the difficulties students experienced during the process are related to the aspects of Habermas' construct of rationality. Therefore, Habermas' construct could be conceived as a tool for determining the difficulties of students in algebraic proving and their origins (Boero & Morselli, 2009; Boero & Planas, 2014; Morselli & Boero, 2011). Furthermore, Radford and Puig (2007) maintained that algebra is the product of a historical-cultural process students encounter at school, and learning algebra does not mean constructing the objects of knowledge; it refers to making sense of the objects. Thus, learning algebra could be considered as a cultural and social process. This view coincides with the novelty our framework would bring to the field of research concerning proving, since the perspective of Habermas' construct of rationality enables us to consider the social and cultural dimensions during the analyses of mathematical activities such as proving, problem-solving and modeling as mentioned by Morselli and Boero (2011).

1.2. The Analysis of Algebraic Proving Process in terms of Habermas' Construct of Rationality

Habermas' construct of rationality could be used as a comprehensive framework to deal with complex educational and cultural problems regarding the teaching and learning of theorems at school (Mariotti et al., 2018). Morselli and Boero (2009) adapted Habermas' construct of rationality to the use of algebraic language in proving. In their study, examples were provided to illustrate how their adaptation of Habermas' construct worked as a tool for the in-depth analysis of the use of algebraic language in the proving process. It was observed that some of the strategic choices of the students did not comply with the aim of the proving process and were not supported by the rigorous checking of inferences. This was interpreted as a combined lack of the teleological and epistemic components of rationality.

Using the adapted version of Habermas' construct, Boero and Morselli (2009) analysed the algebraic proving process of university students. They found that the students made many mistakes during the formalization phase and could not satisfy the modeling requirements of epistemic rationality. It was further revealed that although the students produced the correct expressions, they often could not use these expressions intentionally to achieve their aim. This indicated that they were not able to meet the requirements of teleological rationality.

In another study, Morselli and Boero (2011) analyzed the algebraic proving process in high school students, in students enrolled in the mathematics department, and in elementary mathematics prospective teachers using Habermas' construct of rationality. Their study concluded that strength in epistemic rationality supported teleological and communicative rationality in the proving process. In some cases, students did not realize that the chosen representation did not allow moving towards the goal and thus, they did not change it. The researchers determined that this situation depended on the dominance of teleological rationality without sufficient epistemic control.

Mathematics classroom is a social environment where knowledge was constructed interactively based on the evolving epistemic specificities and where a particular culture of argumentation is intentionally constructed (Boero & Planas, 2014; Zhuang & Conner, 2018). Specific tasks integrated into well-chosen questions appear to be effective in promoting the development of rational behaviors in mathematical activities (Boero et al., 2010; Boero, Guala, & Morselli, 2013; Conner, 2018). Such tasks could help the teacher focus on the epistemic dimension (Zhuang & Conner, 2018) and teleological dimension (Boero et al., 2010; Boero, Guala, & Morselli, 2013), which are often neglected or remain hidden. Considering these advantages, the researchers (Boero et al., 2010; Boero & Planas, 2014; Conner, 2018; Guala & Boero, 2017; Zhuang & Conner, 2018) recommended teachers to plan and conduct mathematical activities considering Habermas' construct of rationality and to analyze students' products based on the rationality components.

Experiences in proof courses and expectations for rigorous proof at university level may lead prospective mathematics teachers to develop a distorted view of proof for middle or high school students (Stylianides, 2007; Suominen, Conner, & Park, 2018). For this reason, prospective mathematics teachers need to engage in proving tasks designed based on the rationality components. In this way, they can realize proving as a rational process and have tendency to design the process of teaching proving and evaluation of students' proof not only based on the characteristics of deductive proof but also within the context of all rationality components. For this, it is firstly required to determine the rationality components at which the prospective teachers are strong or weak. Hence, it will be possible to improve the design of the teaching and learning of proof at university level in a way to reduce the difficulties that prospective teachers experience.

1.3. Aim of the Study

In this study, we aimed to determine the difficulties experienced by prospective mathematics teachers concerning the rationality components in the algebraic proving process and the causes of these difficulties. It is examined the interaction between the rationality components in algebraic proving, and is determined how the phases of production and validation of algebraic expressions were affected by this interaction during the proving process. In this context, the current study sought an answer to the following research question: "What are the difficulties experienced by the prospective mathematics teachers during the proving process in the field of algebra?" The sub-questions that our study aimed to answer are as follows:

1. According to Habermas' construct of rationality, what are the components the prospective mathematics teachers are competent at during the proving process in algebra?
2. According to Habermas' construct of rationality, what are the components the prospective mathematics teachers are incompetent at during the proving process in algebra?
3. How does the dominance or lack of one of the rationality components affect the other rationality components during the proving process in algebra?

It is also aimed to further elaborate on the adapted version of Habermas' construct of rationality by Morselli and Boero (2009), and to add new component(s) to the model in line with the special needs that could emerge during the analysis process. Habermas' construct of rationality is an adequate, general frame to analyze a complex mathematical activity (Mariotti et al., 2018). However, further elaboration on Habermas' construct of rationality has been recommended to improve the approaches of students to conjecturing and proving at school as well as the evaluation of their performances (Boero, 2006, 2017; Boero, Guala, & Morselli, 2013; Boero & Planas, 2014; Cooner, 2018; Guala & Boero, 2017; Morselli & Boero, 2009, 2011). This could be achieved through an in-depth analysis of the relationships between the rationality components (Boero, 2006; Boero & Planas, 2014; Guala & Boero, 2017; Morselli & Boero, 2009, 2011). On the other hand, the requirements of rationality components may depend on different mathematical domains and content (Nardi & Knuth, 2017; Zhuang & Conner, 2018). With this study in algebra, a series of studies, in which students' proving process in different mathematical domains will be analyzed according to Habermas' construct of rationality, has been started.

2. METHODOLOGY

This study is a case study, which is used as an empirical research method in cases where more than one source of proof or data is present (Yin, 2003). The study focused on the proving processes of prospective mathematics teachers in algebra. Four questions that required algebraic proof were prepared. The proving process of each prospective teacher was analyzed within the context of Habermas' construct of rationality based on the coding table presented in Table 1. Based on these codings, the prospective teachers were interviewed in order to reveal the ambiguous parts in their proving process.

2.1. Participants

The study was conducted with the first-year students enrolled in the Mathematics Education Program of a state university in Ankara, Turkey during the spring semester. The ethical approval for the research has been obtained from the Ethics Commission of Hacettepe University (decision number: 35853172/433-354) on January 17, 2017. The participants were selected on a voluntary basis and according to the following criteria:

1. Participants were expected to be first year students. We selected this level since the questions used in the study required a basic level of proving. It was believed that high school mathematics knowledge of the students and the contents of the Analysis I and Abstract Mathematics I courses offered in the fall semester of the first year of the Mathematics Education program would be sufficient for proving the statements. In the fall semester, the freshman students learned the systematic structure of mathematical proof and the didactic structure required for deductive and inductive proving in mathematics courses. Thus, they already had experience to construct a formal proof at basic level and they were competent in proving the given statements in the study.

2. Based on the view that the previous knowledge obtained in a course other than the fall semester mathematics courses could affect the knowledge, theorems and methods students may use in the proving process, the study was carried out with twenty-two students who took the first-year fall semester mathematics courses only once and who did not take any other mathematics courses.

Academic standing was not a criterion when selecting the participants since our aim was to study with the students at multiple levels of academic standing (poor, medium or high). Since the first author of the paper was responsible for teaching the practice sessions of the Analysis II course, she was familiar with the academic standing of the participants.

2.2. Data Collection

Initially, eight questions, which required algebraic proving were prepared. The questions required preliminary knowledge from high school mathematics as well as the contents of the undergraduate mathematics lessons offered in the first semester of Mathematics Education program. Therefore, they required the use of algebraic language in proving at basic level as well as more competent levels.

In order to ensure the reliability of the questions, we reviewed previous studies (Boero, 2006, 2017; Boero, Douek, et al., 2010; Boero, Guala, & Morselli, 2013; Boero & Morselli, 2009; Morselli & Boero, 2009, 2011; Pedemonte, 2007; Pedemonte & Buchbinder, 2011) and various books including questions appropriate to the topic and purpose of our study (Houston, 2009; Nesin, 2010; Velleman, 2006). The opinions of six instructors were obtained to validate the suitability of the questions to the purpose of the research and their mathematical comprehensibility. Based on the opinions of the experts, one question (*Write a continuous function* $f: IR \rightarrow [0,1]$ *so that* $f([0,1]) = \{1\}$ *and* $f(2) = 0$) which was determined to require problem solving rather than proving was excluded from the study. In addition, in line with the expert opinions, each question was written to begin with "*Prove that…*" in order to structure a proving process rather than a problem-solving process. The experts found the variety and number of questions appropriate to the aim of the study and found the level of difficulty suitable for the level of the participants. In order to minimize the possibility of fatigue and distraction that may occur due to the application of the questions in a single session, affecting the performance of the students, the experts suggested grouping the questions according to the difficulty of the proving process and the average time it requires and applying the questions in separate sessions. In order to group the questions in an appropriate manner, the experts and the researchers gathered and formed groups of questions. They were administered to the students in separate sessions, in consecutive days, and the students were given unlimited time in each session.

A pilot study was carried out with nine students, who completed their first year and attended the summer school before the beginning of the new academic year. It was found that two questions provided similar data as the students tried to prove the statements following the same process. Therefore, it was decided to use one of these questions in the main study as this question was seen to present richer data in terms of the analysis of the proving process based on the components of Habermas' construct of rationality. Furthermore, two questions were removed from the study, since the statements given in these questions could not be proven by many students.

As a result, four algebra questions were divided into two groups considering their level of difficulty and were presented to the students in two separate sessions. The students were encouraged to explain their thoughts on the paper during the proving process. There were no time limits during the applications. It was observed that students completed each session in 20 to 30 minutes. After the analysis of the written proving processes, the students were interviewed to make a more detailed analysis of ambigious parts of the written data. The researchers worked together and prepared interview questions before each interview. The students were asked questions such as “*What did you think during the process?*” and “*Can you explain what you did in this question?*”. Furthermore, the students were asked some follow-up questions such as “*How did you understand that the function is differentiable at that point?*”. The interviews lasted between 30 and 40 minutes and they were video-recorded with the consent of the students.

2.3. Data Analysis

In this study, the data were obtained from two sources: written responses that included the proving process of the students and the recording of the interviews. In the analysis of the processes according to the components of Habermas’ construct of rationality, certain criteria were taken into account for each component. During the preparation of the criteria, the studies on the subject were taken into consideration (Boero, 2006, 2017; Boero et al., 2010; Boero, Guala, & Morselli, 2013; Boero &

Morselli, 2009; Boero & Planas, 2014; Conner, 2018; Guala & Boero, 2017; Morselli & Boero, 2009, 2011). The criteria were determined as presented in Table 1.

Table 1.
Criteria Used in the Analysis of Proving Process Based on the Components of Habermas' Construct of Rationality

Rationality Components		Description	Criteria
Epistemic Rationality	MR	Building mathematically accurate and valid algebraic representations (an equation, a function expression, formal definition of a mathematical concept, mathematical rule or theorem)	Justifying the correctness of algebraic formalizations: awareness in building algebraic representation Having coherence in the transition within algebraic representations Justifying the solution method of an equation Interpreting and linking statements within a shared system of knowledge or theory
	SR	Manipulating the rules of signs system correctly and applying the methods of solving equations and inequalities correctly	Calculating correctly (operations, limit calculation, derivative calculation, absolute value calculation, etc.) Using the signs system correctly Substituting a numerical value or an expression into an unknown term in a formula or equation Using formulas considering the distributive property Simplifying the algebraic expression correctly Factorizing the algebraic expression correctly
Teleological Rationality		Choosing and using the knowledge, theorem, definition or rules in accordance with the aim Intertwining with epistemic rationality (choice and justification of the arguments) and with communicative rationality (readers check the production).	Adhering to the algebraic representations: conscious choice and use of algebraic transformations that are useful to the aims of the activity Choosing and using the solution method for an equation consciously Managing the writer-interpreter dynamics consciously
Communicative Rationality		Using the standard notations of mathematics properly, following mathematical rules and criteria to facilitate the reading and manipulation of algebraic expressions; making accurate and valid textual explanations in the process, strictly depending on epistemic and teleological rationality.	Justifying the steps of the proving process: communicating with others (teachers and schoolmates) in an acceptable way using the symbolic language of mathematics or textual explanations. Activating the writer-interpreter dynamics: communicating with oneself by using symbolic language of mathematics or textual explanations related with the steps taken in the process.

The researchers cross-checked the proving processes of the students based on the criteria given in Table 1 in order to validate the results of the analysis. The interview data were transcribed and used to reach an agreement on the results of the analysis on the written proving processes.

3. FINDINGS

In this section, the findings obtained from the data analysis are presented. The difficulties in each question were categorized and the detailed analysis of the proving process of one of the students who experienced such a difficulty was provided. The categories of difficuities were determined among the difficulties which were common in the classroom or which were rarely encountered but were important in terms of the results of the research. Hence, it was revealed how the failures of the students stemmed from the lacks in certain aspects of Habermas' construct of rationality.

In the interview dialogues, the researcher was denoted with the letter "R", while the students were denoted with the letter "S". Furthermore, the answers given by the students during the interview were represented as S_A1, S_A2, etc. The interview data were presented to the reader as direct quotations in the relevant parts of the analysis results.

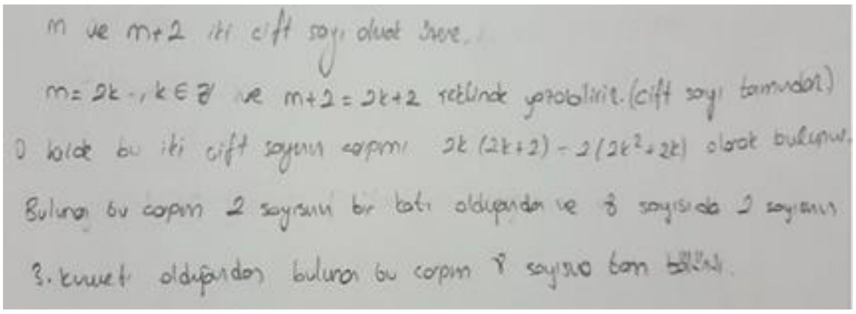
3.1. Difficulties Experienced in the Proving Process of the Expression given in the "Even Numbers" Question

The following question was presented to the students: "*Prove that the product of two consecutive even numbers is divisible by* 8" (Morselli & Boero, 2011). 54% of the students were able to produce a complete and valid proving process. They satisfied the requirements of epistemic and teleological rationality. This situation positively supported the use of formal language and/or textual explanations in the process and hence affected the performance of 68% of the successful students positively in communicative rationality. However, 32% of the successful students, which represents nearly 17% of the class, could not satisfy the requirements of communicative rationality although they satisfied the requirements of epistemic and teleological rationality, and they produced sequences of algebraic calculations with very few and not always appropriate words and formal notations to present their solutions. 46% of the class who failed in the proving process also had problems in communicative rationality and this suggests that in total 63% of the class had problems in communicative rationality in the context of building coherence in the transition within algebraic representations.

In the analysis of the proving processes, it was demonstrated how students' failure was due to the lacks in some aspects of rationality and the effects of these lacks over the other rationality components. The difficulties experienced by the students during the process were grouped under two main categories within the context of the interaction between the rationality components. Section 3.1.1. presents the findings regarding the negative effects of the lacks in teleological rationality on epistemic and communicative rationality. Section 3.1.2. reveals the findings regarding the negative effect of the lacks in epistemic rationality on teleological and communicative rationality. Finally, the findings regarding the rational proving processes which depict the strength of epistemic and teleological rationality are reported in Section 3.1.3.

3.1.1. The negative effect of the lack of teleological rationality on epistemic and communicative rationality

It was observed that 28% of the students failed in the proving process due to the lacks in the teleological rationality. An example for this is given in Figure 1.



m and $m + 2$ are two even numbers. We can write this as $m = 2k$, $k \in Z$ and $m + 2 = 2k + 2$ (based on the definition of even numbers). Then, the product of these two even numbers can be found as $2k(2k + 2) = 2(2k^2 + 2k)$. Since this result is the multiple of 2 and since number 8 is the third power of 2, the result can be divided by 8 exactly.

Figure 1. The lacks in the teleological rationality

This student was able to write the algebraic expressions for the two consecutive even numbers, as 72% of the students did.

> *R: How did you build algebraic expressions for consecutive even numbers?*
> *S_A1: Even numbers must be a multiple of* 2*. Therefore, I wrote the first one of them as* $2k$*, where* k *is an integer number.*
> *R: How did you construct the algebraic expression for the other one?*
> *S_A2: It says consecutive even numbers. So, I need to write the even number that comes just after* $2k$ *algebraically. Since consecutive even numbers increase by* 2*, the second even number becomes* $2k + 2$.

He justified the correctness of the algebraic expressions in the interview by showing awareness about how he built them (S_A1, S_A2). This shows that he satisfied the modeling requirements of the epistemic rationality in terms of building a mathematically accurate and valid algebraic representation. However, he was not able to use the expression $2(2k^2 + 2k)$ properly to reach his aim, as 28% of the students failed in the proving process.

> *R: Can you tell me what you did after you obtained the expression* $2k(2k + 2)$*?*
> *S_A5: I think I can write* $2k(2k + 2) = 2(2k^2 + 2k)$*. What does it mean to be a multiple of* 8*? It means it can be divided by* 8*. What did I write (reads what he wrote in the last line)? I did not understand anything from what I said (Laughs).*
> *R: If you want, you can continue from where you left off and try again.*
> *S_A6: Well, I don't know what to do next. There is* 2 *as a factor, but we also need a* 4*. I don't know how I can get the factor* 4 *from* $2k^2 + 2k$*. I can't continue.*

It was observed that the student was not able to choose and use the appropriate tool required to complete the proof (S_A6); and thus, could not continue the proving process. The student needed to show that the $2k^2 + 2k$ expression is a multiple of 4, but he failed to do so. He should have factorised the expression $(2k + 2)$ instead of expanding $k(2k + 2)$ and should have transformed the expression $2k(2k + 2)$ into $4k(k + 1)$. This indicated that the student lacked teleological rationality in terms of adhering to the algebraic representation by the conscious choice and use of algebraic transformations that are useful to the aims of the activity. This prevented him from having coherence in the transition within algebraic representations (from $2k(2k + 2)$ into

$4k(k+1)$) and from building the expected algebraic representation which is the expression of multiple of 8. It means that the lack in teleological rationality also caused a lack in the modeling requirements of epistemic rationality.

Teleological rationality also intertwines with the communicative rationality within the context of managing the writer-interpreter dynamics by explaining how to transform the expression in accordance with the aim. The problem he experienced in teleological rationality negatively affected communicative rationality, as he was unable to present a complete proof to the reader as seen in Figure 1. Also, he made mathematically insufficient explanations during the interview (S_A5, S_A6). Hence, he could not manage the writer-interpreter dynamics, as 63% of the class and could not satisfy the requirements of communicative rationality.

3.1.2. The negative effect of the lacks in epistemic rationality on teleological and communicative rationality

18% of the students wrote the two consecutive even numbers in terms of k and tried to show that the expression that represents the product of two consecutive even numbers could be divided by 8 by giving numerical values to k. An example of this is given in Figure 2.

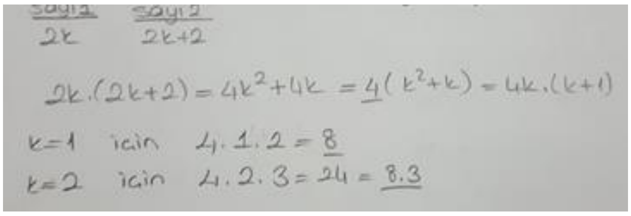


Figure 2. The lacks in the modeling requirements of the epistemic rationality

This student was able to write the algebraic expressions for two consecutive even numbers, as 72% of the students did. The student, who calculated the values of the expression $4k(k+1)$ by substituting numerical values into k and who saw that the results were each time the multiple of 8, ended the process here. Below are some quotations from the interview with this student.

> *R: I see that you gave values to k in the expression $4k(k+1)$. Can you explain what you did from this stage on?*
> *S_A4: I did not give many examples, because it continues like this. Each time the result will be a multiple of 8.*
> *R: Is there any way to guarantee this?*
> *S_A5: Hmm, since the result is a multiple of 8 in these two trials, I thought that it will continue like this.*
> *R: How can we show that the result of this expression for any value of k is a multiple of 8?*
> *S_A6: I understand what you mean. You expect me to show it, I guess algebraically. I mean with algebraic language. However, I cannot continue. I have two very simple terms, k and $k+1$. I can't break these down any further.*

Here, as 18% of the class, the student did not know that if one of the k and $k+1$ represents an even number, the other must represent an odd number (S_A6). She has coherence in the transition within algebraic representations, as she transformed the expression $2k(2k+2)$ into the expression $4k(k+1)$. However, she had deficiencies in the context of interpreting and linking the statements k and $k+1$ within the shared knowledge that states one of the consecutive numbers is always odd when the other one is even or vice versa. It means that she could not satisfy the modeling requirements of epistemic rationality. As a result of lack in modeling requirements of epistemic rationality, she was not able to choose the appropriate tool for her aim and could not complete the process. This indicated that she could not satisfy the requirements of teleological rationality, as 46% of the class.

The lack in presenting the proving process in an acceptable way for the reader (schoolmates and teachers) due to her lacks in epistemic rationality also caused lacks in communicative rationality. She could not manage the writer-interpreter dynamics consciously in the written proving process due to being stuck in the argumentation phase as seen in Figure 2 and in the interview (S_A6). As a result, as 63% of the class, she could not communicate with the reader by using the symbolic language of mathematics or textual explanations, meaning that she could not satisfy the requirements of communicative rationality.

3.1.3. Rational proving processes: the strength of epistemic and teleological rationality

It was observed that some students preferred to use the induction method to prove the expression given in the question, as 24% of the class. An example for such a proving process is given in Figure 3.

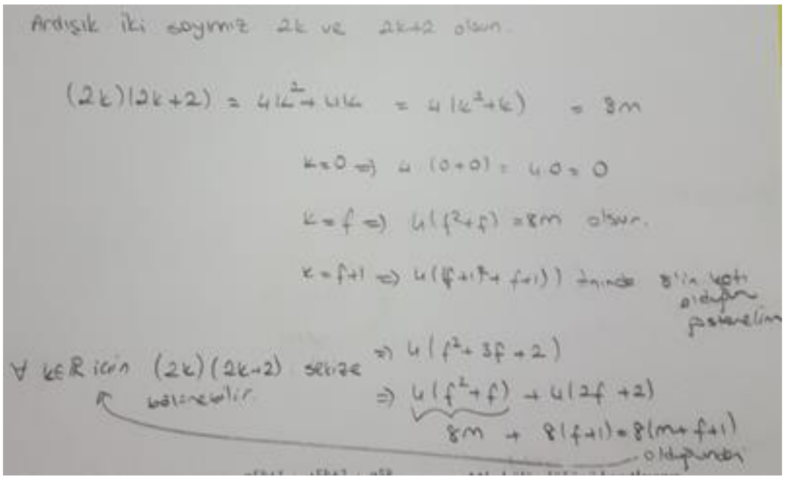


Figure 3. The strength of epistemic and teleological rationality

The student was able to establish correct algebraic expressions and formalizations in the process, as 42% of the students, who preferred this method in the proving process, did. In the interview, the student was able to explain in a mathematically correct and valid way why he represented consecutive even numbers as 2k and 2k + 2. She had coherence in the transition within algebraic representations in the steps she built for $\mathrm{k} = \mathrm{f}$ and $\mathrm{k} = \mathrm{f} + 1$ and justified the correctness of them. In the interview, she also justified that the algrebraic expression she built for $\mathrm{k} = \mathrm{f} + 1$ is a multiple of 8. She justified the whole process by interpreting and linking the inferences she obtained for $\mathrm{k} = 0, \mathrm{k} = \mathrm{f}$ and $\mathrm{k} = \mathrm{f} + 1$ within a shared system of knowledge of the induction method. Therefore, she satisfied the modeling requirements of epistemic rationality.

She also satisfied the systemic requirements of epistemic rationality in terms of the criterion of substituting a numerical value or an expression into an unknown term in the formula (as seen in the step for $\mathrm{k} = 0$ and in the step for $k = f$), using the signs system correctly, using formulas considering the distributive property (as seen in the equation $2\mathrm{k}(2\mathrm{k} + 2) = 4\mathrm{k}^2 + 4\mathrm{k}$), and factorizing the algebraic expressions correctly (as seen in the equations $4\mathrm{k}^2 + 4\mathrm{k} = 4(\mathrm{k}^2 + \mathrm{k})$ and $8\mathrm{m} + 8(\mathrm{f} + 1) = 8(\mathrm{m} + \mathrm{f} + 1)$).

The student used a suitable tool in the proving process by choosing the induction method. She wrote the expression $4(\mathrm{f}^2 + 3\mathrm{f} + 2)$ as $4(\mathrm{f}^2 + \mathrm{f}) + 4(2\mathrm{f} + 2)$. She assumed that the expression $4(\mathrm{f}^2 + \mathrm{f})$ is 8m in the step she structured for $\mathrm{k} = \mathrm{f}$. She also showed that the expression $4(2\mathrm{f} + 2)$ is multiple of 8 by using the common factor 2 in the expression $2\mathrm{f} + 2$ properly. Hence, she transformed the algebraic expression she obtained for $\mathrm{k} = \mathrm{f} + 1$ as $4(\mathrm{f}^2 + 3\mathrm{f} + 2)$ into an expression $8(\mathrm{m} + \mathrm{f} + 1)$, which is multiple of 8. It means that she adhered to the algebraic representations by conscious choice and use of algebraic transformations that are useful to the aims of the activity. Hence, she chose and used the induction method consciously and managed the writer-interpreter dynamics and met the requirements of the teleological component.

The strength of the student in the epistemic and teleological rationality enabled her to justify and present the correct and valid process to the reader. She was able to use the symbolic language of algebra correctly to produce a proof that complied with the formal proving rules. These activated the writer-interpreter dynamics and made it easy to communicate with the reader by symbolic language of mathematics. This indicated that the student satisfied the requirements of communicative rationality in terms of using the symbolic language of mathematics, as 23% of the class did.

3.2. Difficulties in the Proving Process of the Expression given in the "Injective and Surjective Function" Question

The following question was presented to the students: "*Let* $f: A \rightarrow IR$, $f(a) = \frac{2a}{a+1}$ *where* $A = IR \backslash \{-1\}$ *Prove that* f *is injective, but not surjective*" (Houston, 2009; Velleman, 2006). 42% of the students were able to produce a complete and valid proving process. 57% of these students, who were able to produce a complete and valid proving, satisfied the requirements of all of the rationality components. However, 43% of these students could not satisfy the requirements of communicative rationality although they satisfied the requirements of epistemic and teleological rationality. 58% of the class who failed in the proving process had problems in communicative rationality and this suggests that in total, 76% of the class had problems in communicative rationality.

The findings were grouped under three main categories within the context of the interaction between the rationality components. Section 3.2.1. presents the findings regarding the negative effects of the lacks in teleological rationality on communicative rationality. Section 3.2.2. reveals the findings regarding the negative effect of the lacks in the systemic requirements of epistemic rationality on teleological rationality. In Section 3.2.3. the findings regarding the lacks and strengths in communicative rationality are reported.

3.2.1. The negative effect of the lacks in teleological rationality on communicative rationality

It was observed that 32% of the students were able to prove that the function was injective using formal definition; however, they were not able to find the right tool while trying to prove that the function was not surjective. An example of this situation is given in Figure 4.

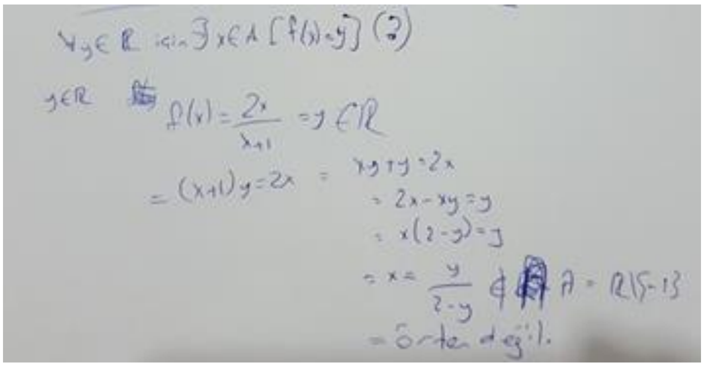


Figure 4. Lacks in teleological rationality

Below are some quotations from the interview with the student.

> *R: In the last step, how did you realize that the function is not surjective?*
> *S_A8: Being surjective means that for every element in the range, there is at least one element in the domain. Therefore, not being surjective means that there is no element in the domain for at least one element in the range. I wanted to show this.*
> *R: Well, have you been able to demonstrate that right now?*
> *S_A9: I guess I couldn't show it here, but I did not know what to do next. I can't go any further from here (shows the expression $x = \frac{y}{2-y}$). That's why I just said there is a problematic y value.*

The student (S_A8), who knew the formal definition of the surjective function (as seen in the first line of the written proving process in Figure 4), was able to correctly build the required formalization for the proof ($f(x) = \frac{2x}{x+1}$). Hence, he had coherence in the transition from formal definition of the surjective function into the algebraic representation of it. He used the distributive property during the calculations (as seen in the transition from $(x + 1)y = 2x$ to $xy + y = 2x$), and factorized the algebraic expressions correctly (as seen in the transition from $2x - xy = y$ to $x(2 - y) = y$). Hence, we can say that he satisfied the systemic requirements of epistemic rationality. He also justified the correctness of the algebraic formalizations by showing his awareness about building them in the interview (S_A8). In this context, he satisfied the modeling requirements of epistemic rationality.

In the last step of the process, he was not able to choose and use the necessary tool to show that the function was not surjective (S_A9). The student could not determine that the $x = \frac{y}{2-y}$ expression is undefined for $y = 2$ and hence, he could not determine that the function $f(x) = \frac{2x}{x+1}$ is not surjective. This indicated that he lacked teleological rationality in the last step. Due to this problem, he was unable to complete the proving process on paper as seen in Figure 4 and made insufficient verbal explanations during the interview process (S_A9). It means that he could not manage the writer-interpreter dynamics consciously, and the lacks in teleological rationality affected communicative rationality negatively. It can be deduced that he failed to satisfy the requirements of communicative rationality, as 76% of the class.

3.2.2. The negative effect of the lacks in the systemic requirements of epistemic rationality on teleological rationality

The student, whose proving process was presented in Figure 5, knew the formal definition of the injective function. However, she was not able to use the equation she built based on the formal definition to achieve her goal, as 21% of the class.

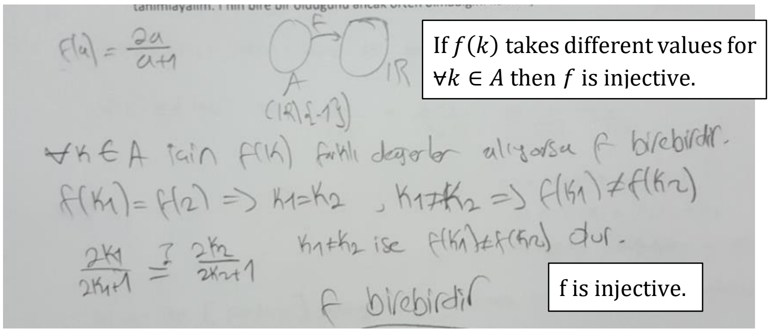


Figure 5. Lacks in the systemic requirements of the epistemic rationality

Below are some quotations from the interview with the student.

R: How did you show that the function is injective?
S_A1: I actually wanted to show it using the formula, but I couldn't manage that. I want to show that when the images are equal, then x (referring to k_1, k_2) are also equal, but I didn't know how to do it. I found the expression confusing.
R: How do you revise it to show that k_1 and k_2 are equal? In the equation, there are fractions on both sides.
S_A2: I can simplify these, but no, it won't work. I can't do it when there is plus or minus between the terms. I do not know how I can do it.

Based on the formal definition of the injective function, the student was able to deduce $f(k_1) = f(k_2) \Rightarrow k_1 = k_2$, as 57% of the class. Hence, she showed coherence in the transition from the definition of the injective function (S_A1) to the formalization $(f(k_1) = f(k_2) \Rightarrow k_1 = k_2)$. Hence, she met the modeling requirements of epistemic rationality.

In the continuation of the process, the student could not organize the expression through cross-multiplication and could not find the equality between k_1 and k_2 (S_A1). It was observed that the student was not able to find the mathematical operation that would enable her to continue the process (S_A2). This indicated that she chose the tool appropriate to her aim $(f(k_1) = f(k_2) \Rightarrow k_1 = k_2)$; however, she could not use this tool properly; therefore, it can be stated that she lacked teleological rationality, 53% of the students. This problem stemmed from the student's inability to perform the required operation during the process. For this reason, it can be argued that she lacked the systemic requirements of epistemic rationality. This prevented her from communicating with others (teachers and schoolmates) in an acceptable way by using the symbolic language of mathematics or textual explanations. Therefore, the student, who presented an incomplete proving process, could not manage the writer-interpreter dynamics consciously in the written proving process as seen in Figure 5 and in the interview (S_A2). It is deduced that she failed to satisfy the requirements of communicative rationality, as 76% of the class.

3.2.3. Lacks and strengths in communicative rationality

It was observed that some students (27% of the class) preferred to prove that the function was surjective using formal definition by finding a counter-example, instead of formal deductive proving. The proving process of one of these students is given in Figure 6.

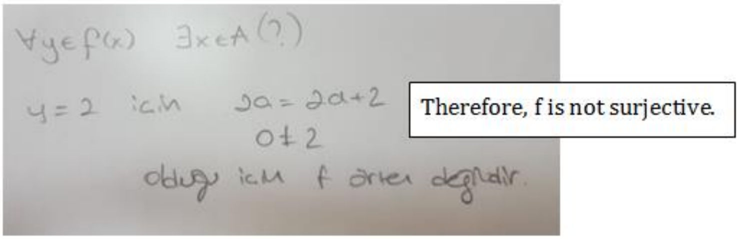


Figure 6. The proving process by the counter-example method

Below are some quotations from the interview with the student.

S_A3: I know that each element in the range should match with at least one element from the domain so that the function is surjective. I already wrote this. But as I did here, this does not work for the $y = 2$ value. For this reason, the function is not surjective.
R: Well, how did you understand that you can't find a suitable x value for $y = 2$?
S_A4: Well, I wanted to try 2 because when $\frac{2a}{a+1} = 2$, the terms $2a$ cancel each other out. I thought that I will reach the result from here and I was right.

The student showed that $\mathrm{y} = 2$ in the range of the function did not match an element from the domain, and thus, the function was not surjective (S_A3). The interview data revealed that he built the equation $\frac{2\mathrm{a}}{\mathrm{a}+1} = 2$ (S_A4). This is a conscious use of the formal definition of surjectivity and serves the aim of the student. Hence, he was able to satisfy the requirements of the teleological rationality, as he chose a suitable proving method for his purpose and was able to use this method correctly.

In the interview, he justified the correctness of the steps in the proving process by communicating with the researcher in an acceptable way. This shows his awareness while building the equation $\frac{2a}{a+1} = 2$ and during the rest of the process. Also, this provides us with evidence that there is coherence in the transition from the formal definition of surjectivity to its algebraic interpretation. This means that he satisfied the modeling requirements of epistemic rationality. He also satisfied the systemic requirements of epistemic rationality, since he made cross-multiplication correctly, used the signs system correctly, and simplified the algebraic expressions properly in the proving process.

Even though he justified the steps of the proving process in the interview and managed the student-interpreter dynamics consciously, he did not show the same performance in his written proving process. He did not write the equation $\frac{2a}{a+1} = 2$ on the paper, and this decreased the clarity and comprehensibility of the proof for the reader. Furthermore, he did not make a textual explanation during the process. Thus, it is understood that the student satisfied the requirements of communicative rationality in the context of verbal communication, but he lacked communicative rationality in terms of managing the writer-interpreter dynamics consciously in the written process, as 42% of the class.

3.3. Difficulties in the Proving Process of the Statement given in the "Divisibility by 11" Question

The following question was presented to the students: "*Prove that for every $k \in IN$, the number $5^{5k+1} + 4^{5k+2} + 3^{5k}$ is divisible by 11*" (Nesin, 2010). 38% of the students were able to produce a complete and valid proving process. However, 53% of the students, who were able to produce a complete and valid proof, could not satisfy the requirements of communicative rationality although they satisfied the requirements of epistemic and teleological rationality. 62% of the class who failed in the proving process had problems in communicative rationality, which suggests that in total, 82% of the class had problems in communicative rationality.

The findings were grouped under three main categories within the context of the interaction between the rationality components. Section 3.3.1. presents the findings regarding the negative effect of the lacks in epistemic and teleological rationality on communicative rationality. Section 3.3.2. reveals the findings regarding the negative effect of the lacks in the systemic requirements of epistemic rationality on the modeling requirements of epistemic rationality. In Section 3.3.3., the findings regarding the lacks and strengths in communicative rationality are reported.

3.3.1. The negative effect of the lacks in epistemic and teleological rationality on communicative rationality

27% of the students used the induction method and the modular arithmetic method together in the process. The proving process of one of these students is given in Figure 7.

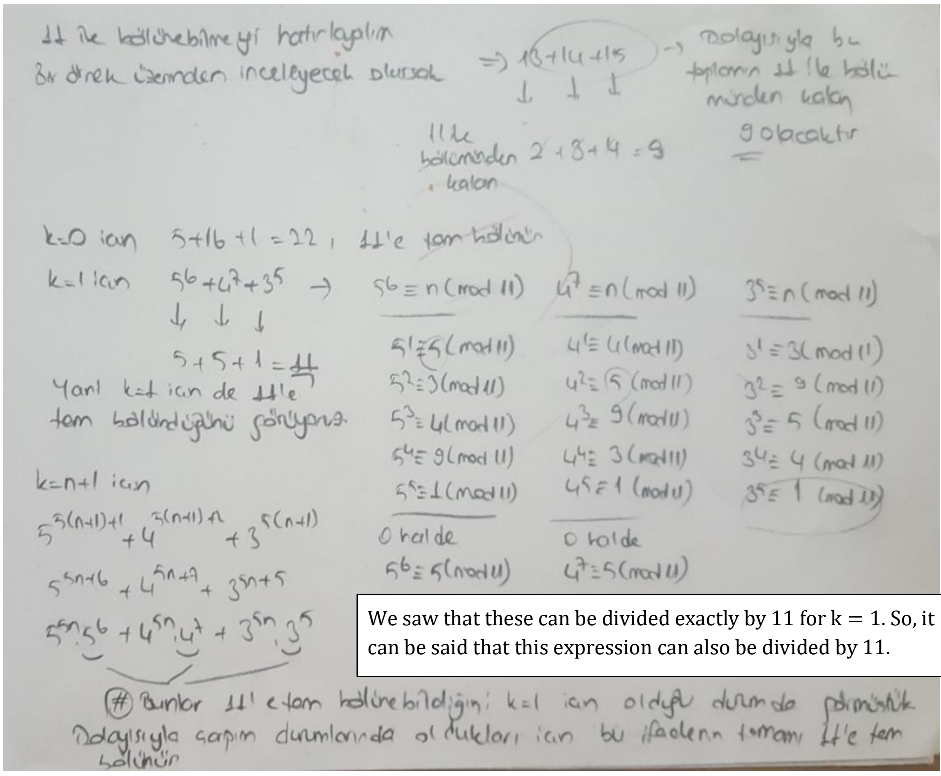


Figure 7. Lacks in epistemic and teleological rationality

The student started the proving process with induction, as 82% of the class did. While constructing the stages of the induction, she tried to show that the result she obtained for $k = 1$ and the expression she obtained for $k = n + 1$ could be divided by 11, using the modular arithmetic method. However, the student skipped the stage in which the expression for $k = n$ was accepted to be true and hence, she could not use the induction method correctly. The explanation of the student in the interview regarding this stage is given below.

> *S_A3: I wanted to prove it using the induction, but I also used modular arithmetic.*
> *R: Could you tell us how you used the induction method?*
> *S_A4: I showed that the statement is true first for $k = 0$ and then for $k = 1$. As I said, I used modular arithmetic here because something confusing emerged for $k = 1$. Then I did it for $k = n + 1$.*
> *R: What does n mean here?*
> *S_A5: Any natural number.*
> *R: Does $n + 1$ have a special meaning?*
> *S_A6: Well, I don't know. I frankly couldn't remember exactly how we did it. Induction was a multi-stage method. I forgot the steps in between.*

The interview data demonstrated that the student did not know how to use the induction method (S_A6), which shows that she has a lack of knowledge about proving with induction and she lacks the modeling requirements of epistemic rationality in terms of justifying the proving method. This lack in the modeling requirements of epistemic rationality caused the student to have difficulty using the method of proving and therefore to be unable to satisfy the requirements of the teleological rationality.

Furthermore, she arranged the expression $5^{5n+6} + 4^{5n+7} + 3^{5n+5}$ for $k = n + 1$ as $5^{5n}5^6 + 4^{5n}4^7 + 3^{5n}3^5$ in the final step of the calculation and she deduced that $5^{5n}5^6 + 4^{5n}4^7 + 3^{5n}3^5$ is divisible by 11, since $5^6 + 4^7 + 3^5$ is a multiple of 11 and $5^{5n} \equiv 1$, $4^{5n} \equiv 1, 3^{5n} \equiv 1 \pmod{11}$. The student chose an incorrect and invalid way to show that the algebraic expression she obtained for k = n + 1 was a multiple of 11, due to the problem she had with the systemic requirements of epistemic rationality, which is related to operation performance. As a result, she had a problem about the modeling requirements of epistemic rationality in terms of interpreting and linking statements within a shared system of operational knowledge. The problem experienced by the student in epistemic rationality also negatively affected the process of using the method she chose to achieve her goal, causing her to have problems with teleological rationality, as 69% of the students who preferred modular arithmetic method in the proving process.

As a result, the lacks in epistemic and teleological rationality prevented her from justifying the steps of the proving process and from offering a valid product to the reader and audience. It means that she could not communicate with others (teachers and schoolmates) in an acceptable way and she could not manage the writer-interpreter dynamics during the proving process (her written explanations are mathematically invalid and there is a missing step and also a faulty step in the written proving process) and the interview (S_A6). Hence, it is deduced that she lacked communicative rationality, as 82% of the class.

3.3.2. The negative effect of the lacks in the systemic requirements of epistemic rationality on the modeling requirements of epistemic rationality

It was observed that 38% of the students tried to prove the statement with the induction method, but did not follow the rules of the simplification of algebraic expressions at the final stage and provided an invalid proof. 63% of these students did not notice and correct their mistakes, and they even defended these mistakes in the interview. The proving process of one of these students is given in Figure 8.

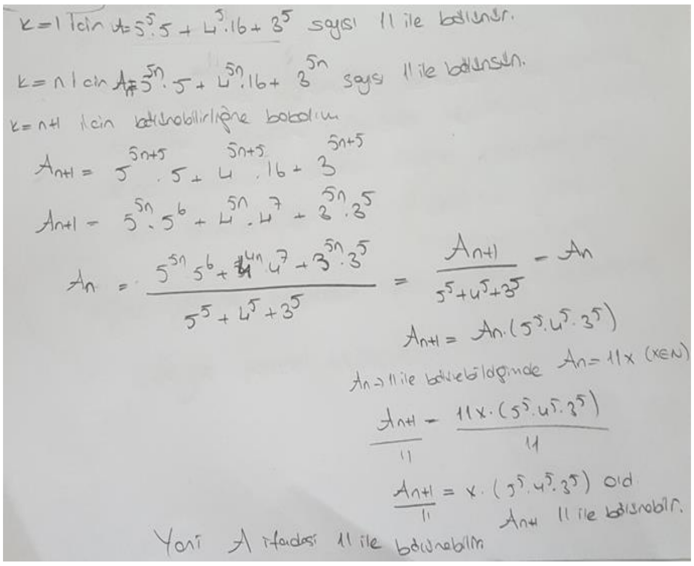


Figure 8. Lacks in the systemic requirements of the epistemic rationality

The student satisfied the requirements of teleological rationality as he chose a valid tool to prove the statement. He calculated the value of the expression for $k = 1$ and found that the result he obtained was a multiple of 11. Then, he assumed that the expression he formulated for $k = n$ is divisible by 11 and he tried to show that the expression he obtained for $k = n + 1$ can be divided by 11. It is seen that he adhered to the induction method and made a conscious choice and use of algebraic transformations that are useful to the application of the method. However, in the final stage, while simplifying the equation he built for A_{n+1}, he acted against the rules of the mathematical operations, which made us think that he has failed due to the lacks in the systemic requirements in this step. Below are the statements of the student during the interview regarding this step.

> *S_A5: I have already found* $5^{5n}5^6 + 4^{5n}4^7 + 3^{5n}3^5$ *for* $k = n + 1$. *So, if I divide it by* $5^5 + 4^5 + 3^5$, *I find exactly what I wrote for* $k = n$ *(showing the expression he found for* A_n*). Hence, I found* A_{n+1} *as* $A_n \cdot (5^5 + 4^5 + 3^5)$. *Well, here since* A_n *is the multiple of* 11, A_{n+1} *can be directly divided by 11.*
> *R: How did you simplify while dividing the expression* A_{n+1} *by* $5^5 + 4^5 + 3^5$*?*
> *S_A6:* 5^6 *and* 5^5 *are simplified and similarly, others too.* 4^7 *and* 4^5 *are simplified and the result becomes* 4^2. 3^5 *and* 3^5 *directly cancel each other out.*

The statements of the student supported our argument that he lacked the systemic requirements of epistemic rationality in terms of simplifying the algebraic expressions correctly. As a result, he constructed a mathematically invalid equation $\frac{A_{n+1}}{5^5+4^5+3^5} = A_n$, where $A_{n+1} = 5^{5n}5^6 + 4^{5n}4^7 + 3^{5n}3^5$ and $A_n = 5^{5n} \cdot 5 + 4^{5n} \cdot 16 + 3^{5n}$. Also, he tried to justify this equation in the interview process, which shows his awareness about building the invalid algebraic representation (S_A6). Hence, it is deduced that the lack in the systemic requirements of epistemic rationality caused him to fail to satisfy the modeling requirements of epistemic rationality.

He presented an invalid proving process to the others (teachers and schoolmates) due to the lacks in epistemic rationality. Furthermore, he tried to justify his invalid steps in the proving process. He managed the writer-interpreter dynamics in an inacceptable way including mathematically invalid steps in the written proving process and mathematically invalid explanations in the interview (S_A5, S_A6). This showed that he failed to satisfy the requirements of communicative rationality, as 82% of the class.

3.3.3. Lacks and strengths in communicative rationality

18% of the students preferred to use the modular arithmetic method entirely during the proving process. All of these students were observed to have succeeded during the process. Figure 9 presents the proving process of a student who preferred the modular arithmetic method.

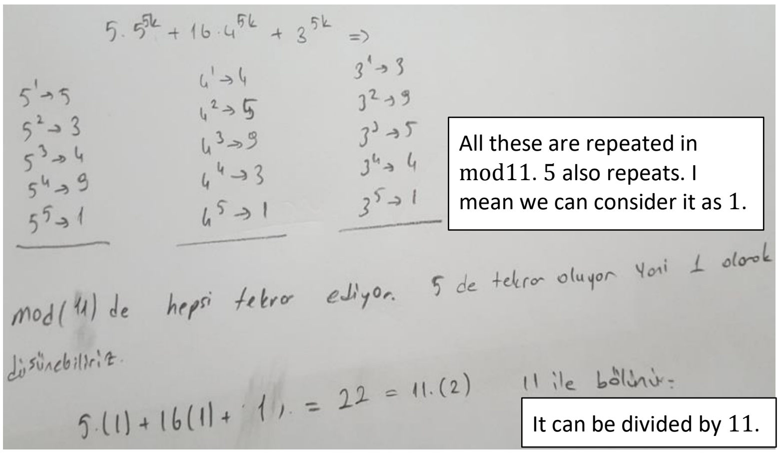


Figure 9. The proving process of one of the students who used modular arithmetic

This student chose a suitable tool (modular arithmetic) to prove the expression and was able to use this tool correctly. He adhered to the method through the conscious choice and use of algebraic transformations that serve the aims of the activity. Hence, he managed the writer-interpreter dynamics consciously. In this context, the student satisfied the requirements of teleological rationality. Below are some quotations from the interview with the student.

> *S_A2: I calculated the powers of* 5. *I divided them by* 11 *each time and calculated the remainder. The remainder for* 5^5 *was* 1. *The same thing happened for* 4^5 *and* 3^5. *The remainder was* 1 *when I divided them by* 11.
> *R: Did you make the calculations yourself or use a calculator in this process?*
> *S_A3: I did it myself, but of course, not in a detailed manner. For example, the first power of* 5 *is itself. The remainder is* 5 *when I divide* 5 *by* 11. *I wrote this. Then the square power of* 5 *is* 25. *When I divide* 25 *by* 11, *the remainder is* 3. *I wrote this as well. Then I stopped calculating power. I continued by multiplying the remainders. For example, I multiplied the remainder of the first and square power to find the remainder of the third power of* 5. *The result was* 15; *I divided* 15 *by* 11 *and I found* 4.

R: Can you explain how this happens?

S_A4: So let's say, for example, $a = 11k + t$ (the student continues by noting what he told on paper). In this case, t is the remainder of the division of what I call a by 11. Let $b = 11z + y$. Let's look at the product of these now. It becomes, $a \times b = (11k + t)(11z + y) = 121kz + 11ky + 11zt + yt$. The first term here is already a multiple of 11, and look, others are too. There is only yt left, and yt is the product of the remainder when the numbers a and b are divided by 11.

He calculated the remainder when the powers of 5, 4 and 3 is divided by 11 and built an iterative system where the remainders were repeated (S_A2, S_A3). This iterative system was the representation that he built and used in the proving process (S_A3). In the interview, he justified the correctness of the algebraic calculations and the iterative system and showed awareness about building them (S_A4). He had coherence in the transition from calculations in the iterative system to the value of the remainder that is obtained when $5^{5k+1} + 4^{5k+2} + 3^{5k}$ is divided by 11 for every $k \in \mathbb{N}$. In this context, he satisfied the modeling requirements of epistemic rationality. Also, he did not make an operational error or act against the mathematical rules during the process, which indicates that he satisfied the systemic requirements of epistemic rationality.

When evaluated in terms of communicative rationality, it was observed that he used an arrow sign instead of the equivalence symbol and did not note down the equivalence classes in symbolic language. He did not include textual explanations so that the reader could easily understand and follow the process. In the final step, he wrote down a sentence; however, this explanation was mathematically meaningless and invalid. The student could not satisfy the requirements of the communicative rationality on paper although he was able to explain and justify the steps in his proving process to the audience during the interview in a correct and understandable way (S_A2, S_A3, S_A4). The strength of the student in the teleological and epistemic rationality positively affected the communicative rationality in terms of verbal communication rather than the use of symbolic language and textual explanation, as seen in 20% of the class.

3.4. Difficulties in the Proving Process of the Expression given in the "Differentiability" Question

The following question was presented to the students: "*Prove that the $f(x) = |x - 1| + 1$ function cannot be differentiated at $x = 1$*" (Houston, 2009; Velleman, 2006). 42% of the students were able to produce a complete and valid proving process. However, 53% of these students could not satisfy the requirements of communicative rationality although they satisfied the requirements of epistemic and teleological rationality. 58% of the class who failed in the proving process had problems in communicative rationality, which suggests that in total, 80% of the class had problems in communicative rationality.

The findings were grouped under three main categories within the context of the interaction between the rationality components. Section 3.4.1. and Section 3.4.2. present the findings regarding the new sub-components under the modeling requirements of epistemic rationality that emerged during the analysis. In Section 3.4.3., the findings regarding the new sub-components in communicative rationality that emerged during the analysis are reported.

3.4.1. The geometric and algebraic representations under the modeling requirements of epistemic rationality: the lacks in algebraic representation

17% of the students calculated the limit of the function when $x \to 1^+$ and $x \to 1^-$ and they stated that the function could be differentiated at the point $x = 1$, if the limit values were equal. Figure 10 shows the proving process of one of these students. The student plotted the graph of the function as 78% of the class. She plotted it correctly and, in the interview, she justified how she drew it. This indicated that she satisfied the modeling requirements of epistemic rationality in terms of building geometric representation, as 74% of the students who preferred to plot the graph of the function. The student also satisfied the systemic requirements of epistemic rationality by correct limit calculations.

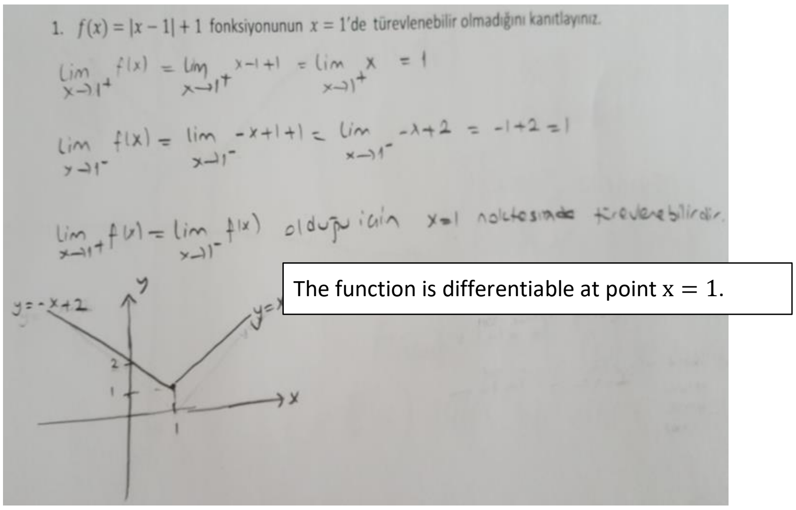


Figure 10. Lacks in the modeling requirements of epistemic rationality

When the right and left limit values of the function were equal at the point $x = 1$, it was understood that the function had a limit value at this point; however, this did not guarantee that the function could be differentiated at that point. Below is an excerpt from the interview with the student.

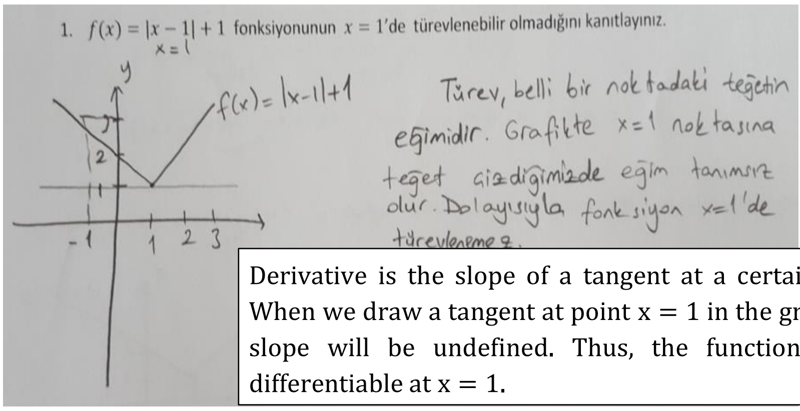
S_A3: I approached to 1 *from right and left and calculated the limit values. The limit values were the same, so the function is differentiable at this point.*
R: But the question says, "Show that the function cannot be differentiated at this point".
S_A4: Oh, I did not notice that. Isn't the function differentiable at $x = 1$*? But how can I show it?*
R: Would you like to try again?
S_A5: Well, I don't know how to do it. I think the function is differentiable. I mean the limit values from the left and the right are the same.

The student justified the invalid formal definition she used (S_A3, S_A5), as 37% of the class. It shows her awareness (S_A3) about building an erroneous formal definition regarding the differentiability of a function at a point. The student realized that she produced an invalid product only when the researcher warned her about the phrase "*… show that it cannot be differentiated*". This means that she had lacks in the modeling requirements of epistemic rationality in terms of building algebraic representation in the proving process. This caused her to consciously choose and use the incorrect tool (if the limit of the function when $x \to 1^+$ and $x \to 1^-$ were equal, then the function could be differentiated at the point $x = 1$). Also, she could not manage the writer-interpreter dynamics when the researcher mentioned that according to the question the function cannot be differentiated at this point (S_A4, S_A5). This means that the problem experienced by the student in the modeling requirements of the epistemic rationality also caused problems in the teleological rationality, as seen in 48% of the class.

The problems in the epistemic rationality prevented the student from building the process in a correct and valid way and from conveying the process to the reader and the audience using valid statements. Furthermore, she justified the invalid steps in her proving process (S_A3) and this prevented the communication with others (teachers and schoolmates) in an acceptable and valid way. In this context, it was observed that the problems experienced in epistemic rationality within the context of building an algebraic representation negatively affected communicative rationality, as 29% of the class.

3.4.2. The geometric and algebraic representations under the modeling requirements of epistemic rationality: the lacks in geometric representation

38% of the students tried to geometrically examine the differentiability of the function at the point $x = 1$ by using the graph of the function. The proving process of one of these students is given in Figure 11.



Derivative is the slope of a tangent at a certain point. When we draw a tangent at point $x = 1$ in the graph, the slope will be undefined. Thus, the function is not differentiable at $x = 1$.

Figure 11. The geometric interpretation of the student about the differentiability of the function at a point

The student tried to examine the differentiability of the function at point $x = 1$ using the slope of the tangent line he drew at this point on the graph, as 85% of the students who preferred to plot the graph of the function did. Actually, this is a valid method, since a unique tangent line must be drawn at point $x = 1$ for the function to be differentiable at that point; however, it is not possible for the given function $f(x) = |x - 1| + 1$ in the statement. It is seen that the student could not use this method in a mathematically acceptable way as seen in the textual explanation given in Figure 11. The statements of the student in the interview regarding this stage are given below.

S_A1: I did it directly from the graph. The graph is already broken at point $x = 1$*. There is always a problem when it is broken. I plotted the tangent at* $x = 1$*. The tangent is drawn exactly horizontal at this point. Its slope is undefined.*
R: How did you understand that?
S_A2: Because it is horizontal. The slope of a horizontal line is undefined.
R: Can you show this?
S_A3: That the slope is undefined?
R: Yes.
S_A4: I don't know exactly how to show it, but as far as I remember, the slope of a horizontal line is undefined.

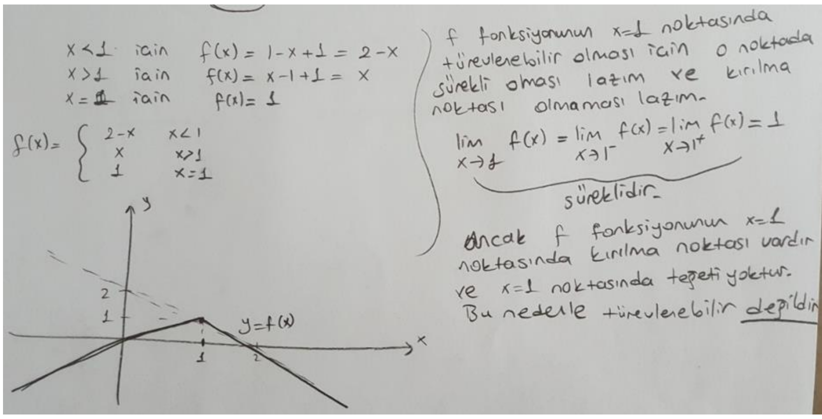
R: What did you do next?
S_A5: The derivative at a point is directly equal to the slope of the tangent at that point. I understood from the graph that the graph of this function breaks at $x = 1$. The tangent is then drawn horizontally at this point. Since the slope of the horizontal line is undefined, I said there is no derivative at this point.

In the interview, the student tried to justify the statement that the slope of a horizontal tangent line is undefined (S_A1, S_A2, S_A5) and showed awareness about this incorrect claim he produced in the writen proving process. He could not interpret and link the geometric representation he built to make a correct deduction about the slope of the tangent line at $x = 1$. This indicates that he could not satisfy the modeling requirements of epistemic rationality within the context of geometric representation. In the interview, he adhered to his invalid claim about the slope of the tangent line at $x = 1$ which he deduced consciously and used it to reach his aim. Hence, it is seen that the problem experienced by the student in the geometric representation negatively affected his teleological rationality, as experienced by 37% of the students who preferred to examine the differentiability of the function at the point x = 1 geometrically.

It seems that he could not manage the writer-interpreter dynamics in the written proving process and in the interview (S_A1, S_A5), because the textual explanation on paper and his phrases in the interview are mathematically invalid. Also, it is noteworthy that he defended his false claims during the interview process. It can be stated that the student had lacks in communicative rationality as he conveyed mathematically false claims to others (teachers and schoolmates). Thus, it can be stated that the problems in epistemic rationality negatively affected the communicative rationality of the student, as 14% of the class.

3.4.3. Lacks and strengths in communicative rationality

It was observed that 34% of the students preferred to make textual explanations in the proving process. Figure 12 presents the proving process of one of these students.



In order for the function to be differentiable at point x = 1, it must be continuous at that point and there must be no breaking point. However, the function has a breaking point at point x = 1 and it does not have a tangent at the same point. Therefore, the function is not differentiable.

Figure 12. Textual explanations in the proving process

The student drew the graph of the function incorrectly on paper, as done by 26% of the class. The statements of the student in the interview regarding this stage are given below.

R: Can you explain how you drew the graph of the function?
S_A1: First, I determined the critical point since it is an absolute value function. I found x-intercept point of the function as $(1, 0)$, which is also the critical point. Then when x is less than 1, when it is 1, and when it is greater than 1, I found the value of the function $f(x) = |x - 1| + 1$.
R: So, how did you draw the graph?
S_A2: I plotted each part of the graph separately and then combined them, but wait a minute! I made a mistake here; it is not true. Give me a second and I will correct it.

In the interview, he realized the mistake and corrected it himself (S_A2). He justified the plotting steps and showed awareness about building geometric representation verbally. Hence, he satisfied the modeling requirements of epistemic rationality in terms of building geometric representations. In the continuation of the proving, he stated that if the function was not continuous then it could not be differentiable at point x = 1 (S_A4).

R: What did you do next?
S_A3: I examined the continuity of the function.
R: Why?
S_A4: Because it cannot be differentiable at $x = 1$ if it is not continuous at that point.
R: How do you know this?
S_A5: I am not sure about the reason, but I know it, because I always use this information in the courses. However, the function is continuous.

R: How did you understand that?
S_A5: I calculated the limit value of the function when x approaches 1 from right and left. The result was the same. Hence, it has a limit value. The limit value at this point is equal to the value the function gets at this point. So, the function is continuous.

Since the function is continuous, the student could not reach his aim. Therefore, he changed his method to prove the statement. The phrases of the student in the interview regarding this stage are given below.

R: So what did you do next?
S_A6: I decided to try the tangent method. This point is the breaking point of the function, so it is not possible to draw a unique tangent at this point. Actually, many tangents can be drawn. The problem is so many different tangents mean so many different slopes. This means that at this point, since the derivative value is the slope of the tangent, it looks like there are many derivative values, but that's not possible. Therefore, the function cannot be differentiable at this point.
R: Well, is it possible for you to show this result using the formal definition of the derivative?
S_A7: To be honest, I don't remember the definition, but since the derivative is the slope of the tangent, I think it is enough to show it like this.

He satisfied the requirements of teleological rationality by choosing and using the proper proving method consciously. In the interview, he interpreted and linked the derivative of a function at a point and the slope of the tangent drawn at that point (S_A6) and ensured coherence in the transition within these concepts verbally. However, he stated that he did not know the formal definition of derivative (S_A7). Hence, he could not build a proper algebraic represention using the formal definition of derivative. Also, he justified the correctness of the definition of continuity verbally (S_A5), however he could not write the formal definition of it and could not build a complete algebraic representation of it on the paper. These show that he had lacks in the modeling requirements of epistemic rationality in terms of building formal definitions, and hence he could not communicate with others (teacher and classmates) using the formal language of mathematics. He made some textual explanations; however, they were not mathematically sufficient due to the lack of linkages between the concepts that would help the reader to understand and follow the process. Furthermore, in the interview, he expressed the transition from continuity to differentiability as a memorized rule (S_A4) and this shows that he also lacked verbal communication in some parts of the proving. This indicated that he had lacks in terms of communicating using formal language, making textual explanations, and verbal explanations, as 42% of the class did.

4. RESULTS, DISCUSSION AND RECOMMENDATIONS

This study aimed to determine the components of rationality that the prospective mathematics teachers were competent and incompetent at, and how dominance or lack of one rationality component affected the other rationality components in algebraic proving. Hence, it is aimed to determine the difficulties the prospective mathematics teachers experienced in the proving process within the context of Habermas' construct of rationality. In this section, the performance of the prospective mathematics teachers in satisfying the requirements of rationality components in algebraic proving was evaluated and the interaction of rationality components with each other was discussed through a comparison with the findings in the literature.

The findings revealed that formalizations and/or interpretations could be accurate; however, they were not goal-oriented (3.1.1., 3.2.1.). Some of the students (3.1.1., 3.2.1.) were able to choose the correct tool for achieving the goal and took all the necessary intermediate steps; however, they were not able to structure the final step. This was believed to have stemmed from the loss of control in epistemic rationality. Boero and Morselli (2009) also revealed that although the students produced the correct expressions, they often could not use these expressions intentionally to achieve their aim and this means that they were not able to meet the requirements of teleological rationality. Morselli and Boero (2011) also determined the high school and university students did not change the proving method due to the dominance of teleological rationality without sufficient epistemic control. It is deduced that teleological rationality should drive formalization under the control of epistemic rationality to achieve the goal.

It was also observed that the formalization and/or interpretations may not be accurate; however, they may be goal-oriented (3.1.2., 3.4.1., 3.4.2.). Some students tried to demonstrate the accuracy of the statement using numerical values only to reach their aims as soon as possible (3.1.2.). Another group of students took steps only to achieve their goal; however, those steps were mathematically invalid (3.4.1., 3.4.2.). It can be deduced that the lacks in epistemic rationality may be resulting from the dominance of teleological rationality which does not include sufficient epistemic control, as mentioned by Morselli and Boero (2011), who encountered the same situation in their study.

Morselli and Boero (2009) found that epistemic rationality of students supported teleological rationality of them in algebraic proving process. In our study, we found that the strength of the students in epistemic rationality supported their teleological rationality (3.1.3.) and vice versa (3.2.3., 3.3.3., 3.4.3.). The performances of students in one of these rationalities was affected by their strength or weakness in the other. Therefore, the performances of students in epistemic and teleological rationality and the interaction of these rationalities in the proving process played a determining role in the successful completion of the proving process.

It was observed that some students demonstrated the accuracy of the statement through numerical examples and they were not able to prove the propositions formally (3.1.2.). As VanSpronsen (2008) stated, these students had difficulty in proving the statement when the accuracy of the statement was clear to them since they believed that there was no need for proof. The products of these students were insufficient based on the requirements of formal proof, which indicated that they lacked communicative rationality in terms of using the formal language of mathematics.

Our study revealed that the lack of epistemic rationality (3.1.2., 3.2.2., 3.3.1., 3.3.2., 3.4.1., 3.4.2.) and teleological rationality in students (3.1.1., 3.2.1., 3.3.1.) affected their performance in communicative rationality negatively. Since the proving process of the students was shaped in line with their performance in epistemic and teleological rationality, it was observed that the communicative rationality was directly nourished by the epistemic and teleological rationality. Morselli and Boero (2009) also mentioned that communicative rationality depended on epistemic rationality and was interrelated to teleological rationality. According to the results of the analysis made within the scope of the "differentiability" question in section 3.4., some students wanted to show that the given function was not differentiable at the given point by using a formal definition; however, they were not able to complete the proving process since they did not know the formal definition about the differentiability of a function at a point (3.4.1.). It was also observed that there were students who approached the differentiability of the function at a point from a geometric perspective and drew the graph of the function correctly; however, they were not able to use the graph correctly and in accordance with the purpose (3.4.2.). This indicated that being able to build a geometric representation in accordance with the given criteria and using this geometric representation is a different ability compared to being able to write the formal definition required to prove the given statement and using this formal definition. In order to make more accurate and detailed analyses, it seemed necessary to add two subcomponents to the modeling requirements of epistemic rationality in the Habermas' construct of rationality. One of these sub-components is the "geometric representation", which is required to assess the ability of the students to build geometric representations according to the criteria given in the question. The "algebraic representation", which is the second sub-component, is required to assess the ability of the students to write the formal definition required to prove the given statement.

It was observed that there were differences in the way students expressed their ideas while proving. The three forms of communication adopted by the students were conveying the proving process using formal language (3.1.3., 3.2.1., 3.2.2., 3.3.1., 3.3.2., 3.4.1.), presenting the process with textual explanations (3.1.1., 3.3.1., 3.3.3, 3.4.2., 3.4.3.) and conveying the process verbally to the audience (specifically see the posteriori interviews in sections 3.2.3 and 3.3.3.). Therefore, the three subcomponents of communicative rationality can be formulated as follows: "symbolic communicative rationality", which is concerned with the correct and appropriate use of the formal language and the notation system of mathematics in the process and the production of formal proof; "textual communicative rationality", which refers to explaining the reasoning process through mathematically correct and valid textual expressions, and presenting a text that can be understood and followed easily by the reader; and "verbal communicative rationality", which involves communicating with the audience verbally through correct and valid mathematical expressions during the interview, and making the process understandable by responding to the questions of the audience with correct and valid answers. There were students who preferred only one of these communication ways, while there were also students who described the process, they experienced by applying multiple communication methods (3.3.1., 3.3.3.).

As Suominen et al. (2018) argue, prospective mathematics teachers develop expectations about their students based on their own experiences. They maintain that due to their experiences in introductory proof courses and the expectations of their instructor about proving, prospective mathematics teachers may develop a distorted view about proving at middle or high school level. As a result, they may not be able to form clear expectations about their own students at middle or high school. For this reason, as also mentioned by Stylianides (2007), prospective teachers need to engage in middle and high school-level proof, ensuring that the level of future proving activities they will design is appropriate for their students at those levels. As argued in this study, in university-level proof teaching courses, prospective teachers should gain experience in middle and high school level proving activities and the products they develop should be evaluated in terms of rationality components. This is because prospective teachers should realize that the process of proving requires acting rationally; and they should design future activities within the framework of rationality components and evaluate students' products taking the rationality components into account, as mentioned by the researchers (Boero et al., 2010; Boero & Planas, 2014; Conner, 2018; Guala & Boero, 2017; Zhuang & Conner, 2018).

Our findings indicated that there is a strong interaction between the components of Habermas' construct of rationality. The performance in one rationality component could have an effect on the other rationality components and lead to failure or success in the proving process as also mentioned by the researchers (Boero et al., 2010; Boero, Guala, & Morselli, 2013; Boero & Morselli, 2009; Boero & Planas, 2014; Conner, 2018; Guala & Boero, 2017; Morselli & Boero, 2009, 2011; Zhuang & Conner, 2018). This demonstrates the usefulness and effectiveness of utilizing Habermas' construct of rationality to analyze the proving processes of students. On the other hand, as Nardi and Knuth (2017) and Zhuang and Conner (2018) maintain, the requirements of the rationality components may be based on the mathematical domain and content. In this study, questions from each of the four subject areas of algebra were studied. The proving processes of students should be analyzed by creating different kinds of questions on algebra topics with different content. Future studies may also analyze students' proving processes in the field of geometry within the scope of Habermas' construct of rationality. The difficulties students experience in proving in the field of geometry within the context of rationality components may be compared with the results obtained in the field of algebra. Hence, the criteria required by rationality components may be updated and the appropriate version of the model for use in both fields of mathematics may be proposed.

Research and Publication Ethics Statement

The authors hereby declare that they have not used any sources other than those listed in the references. The authors further declare that they have not submitted this article at any other journal for publication. The ethical approval for the research has been obtained from the Ethics Commission of Hacettepe University (decision number: 35853172/433-354) on January 17, 2017.

Contribution Rates of Authors to the Article

The authors equally contributed to the article.

Statement of Interest

The authors declare that there is no conflict of interest.

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